1. Find the solution of given differential equations.

1) $y'' + 9y = 4t^2 e^{3t} + 2$

CE: $r^2 + 9 = 0 \Longrightarrow r_1 = 3i,\ r_2 = -3i.$

$y_c = C_1 \cos 3t + C_2 \sin 3t$

Try: $Y(t) = e^{3t}(At^2 + Bt + C) + D.$

$Y'(t) = 3e^{3t}(At^2 + Bt + C) + e^{3t}(2At + B).$

$Y''(t) = 9e^{3t}(At^2 + Bt + C) + 6e^{3t}(2At + B) + 2A \cdot e^{3t}$

Subtitude $Y(t)$ and $Y''(t)$ into the ODE $y'' + 9y = 4t^2 e^{3t} + 2.$

$18e^{3t}(At^2 + Bt + C) + 6e^{3t}(2At + B) + 2Ae^{3t} + 9D = 4t^2 e^{3t} + 2$

Therefore the coefficient should satisfy the follow equations.

$$\begin{cases} 18A = 4 \\ 18B + 12A = 0 \\ 18C + 6B + 2A = 0 \\ 9D = 2 \end{cases} \Longrightarrow \begin{cases} A = \frac{2}{9} \\ B = -\frac{4}{27} \\ C = \frac{2}{81} \\ D = \frac{2}{9} \end{cases}$$

The solution is $y(t) = C_1 \cos 3t + C_2 \sin 3t + (\frac{2}{9}t^2 - \frac{4}{27}t + \frac{2}{81})e^{3t} + \frac{2}{9}$

2 $y'' - 2y' - 3y = -te^{-t}$

Solution:

$C.E: r^2 - 2r - 3 = 0 \Rightarrow r_1 = 3 \quad r_2 = -1$

$y_c = C_1 e^{-t} + C_2 e^{3t}$

Try $Y(t) = t(At+B)e^{-t} = (At^2 + Bt)e^{-t}$

$Y'(t) = -e^{-t}(At^2+Bt) + e^{-t}(2At+B)$

$Y''(t) = e^{-t}(At^2+Bt) - e^{-t}(2At+B) + e^{-t} \cdot 2A - e^{-t}(2At+B)$

$= e^{-t}(At^2+Bt) - 2e^{-t}(2At+B) + A2e^{-t}$

$Y''(t) - 2Y'(t) - 3y = (At^2+Bt)e^{-t} - 2e^{-t}(2At+B) + 2Ae^{-t} - 2[-e^{-t}(At^2$
$+Bt) + e^{-t}(2At+B)] - 3(At^2+Bt)e^{-t} = -te^{-t}$

$$\Rightarrow \begin{cases} A + 2A - 3A = 0 \\ \cancel{B} - 4A + \cancel{2B} - 4A - \cancel{3B} = -1 \\ -2B - 2B + 2A = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{8} \\ B = \frac{1}{16} \end{cases}$$

$y(t) = C_1 e^{-t} + C_2 e^{3t} + \left(\frac{1}{8}t^2 + \frac{1}{16}t\right)e^{-t}$

3. $y'' + 4y = \cos 2t \qquad y(0)=2, \; y'(0)=1$

$C.E: \; r^2 + 4 = 0 \qquad r_1 = -2i. \quad r_2 = 2i.$

$$y_c = c_1 \cos 2t + c_2 \sin 2t$$

Try:

$$Y(t) = At \sin 2t + Bt \cos 2t$$

$$Y'(t) = A \sin 2t + 2At \cos 2t + B \cos 2t + Bt \sin 2t\,(-2)$$

$$= (A - 2Bt) \sin 2t + (2At + B) \cos 2t$$

$$Y''(t) = (-2B) \sin 2t + (A - 2Bt) \cdot 2 \cos 2t + 2A \cos 2t - 2(2At + B) \sin 2t$$

$$= (-2B - 4At - 2B) \sin 2t + (2A - 4Bt + 2A) \cos 2t$$

$$(-2B \cancel{-4At} - 2B \cancel{+4tA}) \sin 2t + (\cancel{4Bt} + 2A \cancel{-4Bt} + 2A) \cos 2t$$

$$= \cos 2t$$

$$\begin{cases} -4B = 0 \\ 4A = 1 \end{cases} \Rightarrow \begin{cases} B = 0 \\ A = \frac{1}{4} \end{cases}$$

$$Y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4} t \sin 2t$$

$$y(0) = C_1 = 2$$

$$y'(0) = 2C_2 = 1 \Rightarrow C_2 = 1/2$$

$$y(t) = 2 \cos 2t + \frac{1}{2} \sin 2t + \frac{1}{4} t \sin 2t$$

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2. Without solving the equation, determine a suitable form for particular solution $Y(t)$.

1) $y''+3y'+2y=e^{-t}(t+3)\sin 2t+e^{t}\sin 3t+10$

CE: $r^2+3r+2=0 \quad r_1=-1 \cdot r_2=-2$

$y_c=C_1e^{-t}+C_2e^{-2t}$

$Y(t)=e^{-t}(At+B)(C\sin 2t+D\cos 2t)+e^{t}(E\sin 3t+F\cos 3t)$
$+G$.

2) $y''+2y'=2t+\cos t+e^{-2t}t^3$.

CE: $r^2+2r=0 \quad r_1=0 \; r_2=-2$

$y_c=C_1+C_2e^{-2t}$

$Y(t)=t(At+B)+C\sin t+D\cos t+te^{-2t}(Et^3+Ft^2+Gt+H)$

3. 1) $t^2y''-t(t+2)y'+(t+2)y=2t^3, \quad t>0 \quad y_1(t)=t \quad y_2(t)=te^{t}$

$y''-\left(\frac{t+2}{t}\right)y'+\left(\frac{t+2}{t^2}\right)y=2t$

$W=\begin{vmatrix} t & te^{t} \\ 1 & (t+1)e^{t} \end{vmatrix}=t^2e^{-t}$

$$Y(t) = -t\int \frac{te^{t}\cdot 2t}{t^2e^t}\,dt + te^t\int \frac{t\cdot 2t}{t^2e^t}\,dt = -2t^2$$

$$y(t) = C_1t + C_2te^t - 2t^2$$

21. $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t>0, \quad y_1(t) = 1+t \quad y_2(t) = e^t$

$$y'' - \left(\frac{1+t}{t}\right)y' + \frac{y}{t} = te^{2t}$$

$$W = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = te^t$$

$$Y(t) = -(1+t)\int \frac{e^t\cdot te^{2t}}{te^t}\,dt + e^t\int \frac{(1+t)te^{2t}}{te^t}\,dt$$
$$= \left(\frac{t-1}{2}\right)e^{2t}$$

$$y(t) = C_1(1+t) + C_2e^t + \frac{1}{2}(t-1)e^{2t}$$

4(a) $t^2y'' - 2y = 0, \quad t>0 \quad y_1(t) = t^2$. Let $y = v\cdot y_1$

$$\Rightarrow y_1v'' + (2y_1' - 0\cdot y_1)v' = 0$$
$$t^2v'' + (2\cdot 2t - 0)v' = 0 \Rightarrow (v')' + \frac{4}{t}v' = 0$$
$$\Rightarrow v' = C_1t^{-4} \Rightarrow v = C_2t^{-3} + C_3$$
$$y = vy_1 = (C_2t^{-3} + C_3)t^2 = C_2t^{-1} + C_3t^2$$

So $y_2 = t^{-1}$

b) Using the method of variation of parameters to solve the following equation.

$$t^2 y'' - 2y = 3t^2 - 1$$

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}$$

$$W = \begin{vmatrix} t^2 & \frac{1}{t} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3$$

$$Y(t) = -t^2 \int \frac{\frac{1}{t}(3 - \frac{1}{t^2})}{-3} dt + \frac{1}{t} \int \frac{t^2(3 - \frac{1}{t^2})}{-3} dt$$

$$= \frac{t^2}{3} \int (\frac{3}{t} - t^{-3}) dt - \frac{1}{3t} \int (3t^2 - 1) dt$$

$$= \frac{t^2}{3} (3\ln t + \frac{t^{-2}}{2}) - \frac{1}{3t} (t^3 - t)$$

$$= t^2 \ln t + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3} = t^2 \ln t - \frac{t^2}{3} + \frac{1}{2}$$

$$y(t) = C_1 t^2 + C_2 t^{-1} + t^2 \ln t + \frac{1}{2}$$

5 1). $12t - 3, \quad t^2 + 1, \quad 2t^2 - t$

$$W = \begin{vmatrix} 12t-3 & t^2+1 & 2t^2-t \\ 12 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 12t-3 & 2t^2-t \\ 12 & 4t-1 \end{vmatrix} + 4 \begin{vmatrix} 12t-3 & t^2+1 \\ 12 & 2t \end{vmatrix}$$

$$= -2(48t^2 - 12t - 12t + 3 - 24t^2 + 12t) + 4(24t^2 - 6t - 12t^2 - 12) = -54 \neq 0$$

$\Rightarrow 12t-3,\ t^2+1,\ 2t^2-t$ are linear independent.

2) $1,\ \cos t,\ \sin t$

$$W = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = \sin^2 t + \cos^2 t = 1 \neq 0$$

$\Rightarrow 1,\ \cos t$ and $\sin t$ are linear independent.

6 1) $y''' - y'' - y' + y = 0$

CE: $r^3 - r^2 - r + 1 = 0$

$\Rightarrow r^2(r-1) - (r-1) = 0$

$\Rightarrow (r^2-1)(r-1) = 0$

$\Rightarrow (r+1)(r-1)^2 = 0$

$y_c = C_1 e^{-t} + C_2 e^t + C_3 t e^t$

2). $y^{(4)} - y'' = 0$

C.E: $r^4 - r^2 = 0 \Rightarrow r^2(r^2-1) = 0 \Rightarrow r^2(r+1)(r-1) = 0$

$y_c = C_1 + C_2 t + C_3 e^{-t} + C_4 e^t$

3) $y''' - y'' + y' - y = 2e^t + 3$

CE: $r^3 - r^2 + r - 1 = 0$

$\Rightarrow r^2(r-1) + (r-1) = 0$

$\Rightarrow (r^2+1)(r-1) = 0$

$\Rightarrow (r+i)(r-i)(r-1) = 0$

$y_c = C_1 e^t + C_2 \cos t + C_3 \sin t$

$Y(t) = Ate^t + B$

$Y'(t) = e^t(At + A)$

$Y''(t) = e^t(2A + At)$

$Y'''(t) = e^t(3A + At)$

$\Rightarrow e^t(3A + \cancel{At} - 2A - \cancel{At} + \cancel{At} + A - \cancel{At}) - B = 2e^t + 3$

$2Ae^t - B = 2e^t + 3$

$\Rightarrow A = 1$

$B = -3$

$\Rightarrow y(t) = C_1 e^t + C_2 \cos t + C_3 \sin t + te^t - 3$

4) $y^{(4)} - y = 3t + \sin t$

CE: $r^4 - 1 = 0 \Rightarrow (r^2+1)(r^2-1) = 0 \Rightarrow (r+i)(r-i)(r+1)(r-1) = 0$

$y_c = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

Try $Y(t) = At + B + t(C\sin t + D\cos t)$

$Y'(t) = A + C\sin t + D\cos t + t(C\cos t - D\sin t)$

$Y''(t) = C\cos t - D\sin t + C\cos t - D\sin t + t(-C\sin t - D\cos t)$

$Y'''(t) = -3C\sin t - 3D\cos t + t(-C\cos t + D\sin t)$

$Y''''(t) = -4C\cos t + 4D\sin t + t(C\sin t + D\cos t)$

$y'''' - y = 3t + \sin t$

$-4C\cos t + 4D\sin t + t(C\sin t + D\cos t) - At - B$

$-t(C\sin t + D\cos t) = 3t + \sin t$

$\Rightarrow \quad A = -3$

$B = 0$

$C = 0$

$D = 1/4$

$y(t) = C_1 e^{t} + C_2 e^{-t} + C_3 \cos t + C_4 \sin t - 3t + \frac{t}{4}\cos t$