Mathematics 2260: ODE (I)

Solution to Assignment 2

1. Find the solutions by using substitution method

$$1)\ y' = (x+2y)/(2x+y) \quad 2)\ \ (y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0$$

1)5': Rewrite the equation as

$$\frac{dy}{dx} = \frac{1+\frac{2y}{x}}{2+\frac{y}{x}}.$$

Assume $v = \frac{y}{x}$. Then $y' = \frac{dy}{dx} = v + xv'$ and the equation becomes

$$v + xv' = \frac{1+2v}{2+v}$$

or

$$xv' = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v}.$$

So we have

$$\frac{2+v}{1-v^2}dv = \frac{1}{x}dx$$

Since

$$\frac{2+v}{1-v^2} = \frac{\frac{1}{2}}{1+v} + \frac{\frac{3}{2}}{1-v},$$

we have

$$\int \left(\frac{\frac{1}{2}}{1+v} + \frac{\frac{3}{2}}{1-v} \right) dv = \int \frac{1}{x}dx$$

and

$$\frac{1}{2}\ln|1+v| - \frac{3}{2}\ln|1-v| = \ln|x| + C.$$

Returning to y, we have

$$\frac{1}{2}\ln|1+\frac{y}{x}| - \frac{3}{2}\ln|1-\frac{y}{x}| = \ln|x| + C.$$

2)5':rewrite the equation as

$$\frac{dy}{dx} = -\frac{y^4 - 2x^3y}{x^4 - 2xy^3}$$

and

$$\frac{dy}{dx} = -\frac{\left(\frac{y}{x}\right)^4 - 2\frac{y}{x}}{1 - 2\left(\frac{y}{x}\right)^3}$$

assume that

$$v = \frac{y}{x}.$$

Then we have

$$y' = v + xv'$$

and

$$xv' = -\frac{v^4 - 2v}{1-2v^3} - v = \frac{v+v^4}{1-2v^3}.$$

So

$$\frac{1-2v^3}{v+v^4}dv = \frac{1}{x}dx$$

or

$$\frac{1}{v} + \frac{-3v^2}{1+v^3} = \frac{1}{x}dx.$$

taking integration both sides gives

$$\ln|v| - \ln|1+v^3| = \ln|x| + C$$

or

$$\frac{v}{1+v^3} = cx, \quad c = e^C.$$

returning to y, we have

$$\frac{y/x}{1+(y/x)^3} = cx.$$

2. 1) Assume that $v = y'$, then we obtain

$$v' + t^{-1}v = 0.$$

Solve this equation gives

$$v = c_1 t^{-1}.$$

Since $v = y'$, we have

$$\begin{aligned} y &= \int v(t)dt \\ &= c_1 \int t^{-1}dt + c_2 \\ &= c_1 \ln|t| + c_2. \end{aligned}$$

2) rewrite the equation as $t^2\frac{y'}{y^3} + 2t\frac{1}{y^2} = 1$. Set $v = \frac{1}{y^2}, v' = -2\frac{y'}{y^3}$, So we have

$$t^2v' - 4tv = -2.$$

Solve the above equation. $y = \pm[5t/(2+5ct^5)]^{1/2}$.
3).From Bernoulli equation or seperation of variable, we have $y = \frac{re^{rx}}{cr+ke^{rx}}$.

3. suppose that a certain population obeys the logistic equation $dy/dt = ry[1-y/K]$.

(a) If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ *per year*.
(b)If $y_0 = \alpha K$, find the time T at which $y(T) = \beta K$, where $0 < \alpha, \beta < 1$.
Solution:(a) 4': Using

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + [1-y_0/K]e^{-rt}},$$

we have

$$e^{-rt} = \frac{y_0/K[1-y/K]}{y/K[1-y_0/K]}$$

or

$$t = \frac{-1}{r}\ln\frac{y_0/K[1-y/K]}{y/K[1-y_0/K]}. \tag{1}$$

Seting $y = 2y_0$,

$$\tau = \frac{-1}{r} \ln \frac{1}{4} = 55.45 years$$

(b) 4': in (1), set $y_0/K = \alpha$ and $y/K = \beta$. as a result, we obtain

$$T = -\frac{1}{r} \ln |\frac{\alpha(1-\beta)}{\beta(1-\alpha)}|.$$

4. $y' = ry - H$. Re-write $y' - ry = -H$. $\mu = e^{-rt}$. $y(t) = \frac{H}{r} + Ce^{rt}$.

$$y(0) = 10 \to C = 10 - \frac{H}{r}$$

So the solution is $y(t) = \frac{H}{r} + (10 - \frac{H}{r})e^{rt}$.

1)$H = 0$. $y(t) = 10e^{rt}$. $y(3) = 45 \to 45 = 10e^{3r} \to r = \frac{\ln 4.5}{3} = 0.501$.
2)$17 = \frac{H}{0.501} + (10 - \frac{H}{0.486}) \times 4.5 \to H = 4.01$.
3) y(10)=$\frac{4.01}{0.501} + (10 - \frac{4.01}{0.501})e^{10\times\frac{\ln 4.5}{3}} = 309$.

5. Determine (with out solving the problem) an interval in which the solution of the given initial value problem is certain to exist:

$$t(t-4)y' + y = 2t, \ y(2) = 1.$$

Solution(4'): Divided both side by $t(t-4)$, we have

$$y' + \frac{1}{t(t-4)}y = \frac{2}{t-4}.$$

Since $p(t) = \frac{1}{t(t-4)}$ is not continuous at $t = 0$ and $t = 4$, and $g(t) = \frac{2}{t-4}$ is not continuous at $t = 4$ and the initial point t_0 is 2, we have that in the interval **(0,4),** the solution is certain to exist.

6. Determine whether or not each of the following equation is exact. If it is exact, find the general solution; otherwise, find an integrating factor and solve it:

1) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$ 2) $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0$

3) $\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$ 4) $ydx + (2xy - e^{-2y})dy = 0$

Solution 1) 4': $M = 2xy^2 + 2y, N = 2x^2y + 2x$, $M_y = 4xy + 2$ and $N_x = 4xy + 2$. Thus

$$M_y = N_x, \ the\ equation\ is\ exact.$$

$$\begin{aligned}\Psi(x,y) &= \int M dx + h(y) \\ &= x^2y^2 + 2xy + h(y).\end{aligned}$$

By $\Psi_y = N$, we have

$$h'(y) = 0$$

and $h(y) = c$. So the solution is determined by

$$x^2y^2 + 2xy = C.$$

2) 4':

$$M = (e^x \sin y - 2y \sin x), \; N = (e^x \cos y + 2\cos x)$$

So

$$M_y = e^x \cos y - 2\sin x = N_x = e^x \cos y - 2\sin x.$$

The equation is exact and

$$\begin{aligned}\Psi(x,y) &= \int M dx + h(y) \\ &= e^x \sin y + 2y\cos x + h(y)\end{aligned}$$

By $\Psi_y = N$, we have $h'(y) = 0$ and $h(y) = c$.

So the solution is determined by

$$e^x \sin y + 2y \cos x = C.$$

3) 4': Rewrite the equation as

$$(ax + by)dx + (bx + cy)dy = 0$$

where $M = ax + by, N = bx + cy$.

$$M_y = b = N_x.$$

So this equation is exact.

$$\begin{aligned}\Psi(x,y) &= \int M dx + h(y) \\ &= \frac{1}{2}ax^2 + bxy + h(y).\end{aligned}$$

Using $\Psi_y = N$ gives

$$h'(y) = cy$$

and

$$h(y) = \frac{cy^2}{2}.$$

The solution is determined by

$$\frac{1}{2}ax^2 + bxy + \frac{cy^2}{2} = C.$$

4) 4':

$$M = y, \quad N = 2xy - e^{-2y}$$

and

$$M_y = 1 \neq N_x = 2y.$$

It is not an exact equation.

Assume that μ is a function of y. then

$$\mu'_y = -\mu\frac{M_y - N_x}{M} = -\mu\frac{1-2y}{y} = \mu(2 - \frac{1}{y}).$$

Solving this equation to take

$$\mu = \frac{1}{y}e^{2y}.$$

Then the new equation is

$$e^{2y}dx + (2xe^{2y} - \frac{1}{y})dy = 0.$$

We have

$$\begin{aligned}\Psi &= \int e^{2y} dx + h(y) \\ &= xe^{2y} + h(y).\end{aligned}$$

Using $\Psi_y = N$, we have

$$h'(y) = -\frac{1}{y}$$

and take $h(y) = -\ln y$. So the solution is determined by

$$xe^{2y} - \ln y = C.$$

7. Determine the longest interval in which the given value problem is certain to have twice differentiable solution. Do not attempt to find the solution.

$$1)\ (x-3)y'' + xy' + (\ln|x|)y = 0,\ \ y(1) = 0, y'(1) = 1$$
$$2)\ (x-2)y'' + y' + (x-2)(\tan x)y = 0,\ \ y(3) = 1, y'(3) = 2.$$

Solution 1)3': re-write the equation as

$$y'' + \frac{x}{(x-3)}y' + \frac{1}{(x-3)}(\ln|x|)y = 0.$$

Obviously $p(t) = \frac{x}{(x-3)}$ is not continuous at $x = 3$ and $q(t) = \frac{1}{(x-3)}(\ln|x|)$ is not continuous at $x = 3$ and $x = 0$. The initial point t_0 is 1. So the lonest interval is (0,3).

2) 3': rewrite the equation:

$$y'' + \frac{1}{(x-2)}y' + (\tan x)y = 0$$

$p(t) = \frac{1}{x-2}$ is not continuous at $x = 2$. Since $\tan x$ is not continuous at $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots.$ and it is continuous in the interval $(\pi/2,\ \frac{3}{2}\pi)$, and the initial point t_0 is 3, we have that the longest interval is $(2, \frac{3}{2}\pi)$.