Mathematics 3260: ODE (I)

Assignment 1, **Solution**

1.Determine the order of the following equations; also state whether the equation is linear or nonlinear.

$$(a)\ \frac{d^4y}{dt^4} + 3t^4\frac{d^2y}{dt^2} + y = \sin t^4; \qquad (b)\ \frac{d^2y}{dt^2} + \cos(t+y) = t^2$$

$$(c)\ u_{xx} + u_y + u_{yyyx} = 0; \qquad (d)\ u_t + uu_x = 1 + u_{xxx}$$

Solution (a)order 4;linear (b)order 2;nonlinear(c)order 4;linear (d) order 3;non-linear

2. Verify that $y_1 = e^{-3t}$ and $y_2 = e^t$ are the solutioms of the following equation

$$y'' + 2y' - 3y = 0, t > 0.$$

Substitute $y_1 = e^{-3t}$ to the equation $y'' - 2y' - 3y = 0$ to have

$$(e^{-3t})'' + 2(e^{-3t})' - 3e^{-3t} = 0.$$

Do the same for $y_2 = e^t$.

3. Consider a population p of field mice that grows at a rate propotional to the current population:

$$\frac{dp}{dt} = rp \quad \text{where } r \text{ is a constant.}$$

(a) Assume that $p(0) = p_0$. Find the rate r so that the population doubles in 30 days.

(b)Find r if the population doubles in N days.

Solution (a) The general solution to the DE is $p(t) = Ce^{rt}$. Using $p(0) = p_0$ to have $C = p_0$. The solution is $p(t) = p_0e^{rt}$. Using $p(30) = 2p_0$ gives

$$2p_0 = p_0e^{30r}$$

or $r = \frac{\ln 2}{30}$.
(b) $r = \frac{\ln 2}{N}$

4. Find the general solution of the following problem and use it to determine how the solutions behave as $t \to \infty$:

$$(a)\ y' + 3y = 1 + e^{-2t} \quad (b)\ y' - 2y = t^2e^{2t} \quad (c) y' + y = te^{-t} + 1$$

(a) $\mu(t) = e^{3t}$. y(t)=$\frac{1}{3} + e^{-2t} + Ce^{-3t}$. Therefore, y$\to \frac{1}{3}$ as $t \to \infty$.
(b)$\mu = e^{-2t}$. y(t)= $t^3e^{2t}/3 + Ce^{2t}$. All solution increase at an exponential rate.
(c) $\mu = e^{2t}$. $y(t) = t^2e^{-t}/2 + 1 + Ce^{-t}$. All solution converge to constant 1.

5. Solve the equation: $t^3y' + 4t^2y = e^{-t}, y(-1) = 0, t < 0$ by the method of integrating factor.

Solution: rewrite

$$y' + \frac{4}{t}y = t^{-3}e^{-t}$$

$\mu = e^{\int \frac{4}{t}dt} = t^4$. y(t)= $-\frac{(t+1)e^{-t}+c}{t^4}$. $y(-1) = 0 \Rightarrow c = 0$.

6. Find the solutions to the following initial problem

(a) $y' + 2y = te^{-2t},\ y(1) = 0$; (b) $y' + \frac{2}{t}y = \cos(t)/t^2,\ y(\pi) = 0$

(c) $y' - 2y = e^{2t},\ y(0) = 2, t > 0$.

Solution (a) $\mu = e^{2t}$; $y(t) = t^2e^{-2t}/2 + ce^{-2t}; c = -\frac{1}{2}$.
(b)$\mu = t^2$. y(t)=$\frac{\sin t}{t^2}$.
(c)$\mu = e^{-2t}$. $y(t) = (t+2)e^{2t}$.

7. Find the general solutions of the following equations by the method of seperation of variables.

(a) $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^y}$ (b) $y' = 2\sin(2x)/(3+2y)$, (c) $xy' = (1-y^2)$

(d) $y' = 2x/(1+2y),\ y(2) = 0$.

Solution (a)(y+e^y)$dy = (x - e^x)dx$;

$$\int (y + e^y)dy = \int (x - e^x)dx \Rightarrow y^2/2 + e^y = \frac{1}{2}x^2 + e^{-x} + C$$

(b) 3y+y^2 $= -\cos 2x + C$
(c) $\frac{dy}{1-y^2} = \frac{1}{x}dx \Rightarrow \frac{1}{2}\frac{dy}{1+y} - \frac{1}{2}\frac{dy}{1-y} = \frac{1}{x}dx \Rightarrow \frac{1}{2}\ln|1+y| - \frac{1}{2}\ln|1-y| = \ln x + C$
(d)$y + y^2 = x^2 + C \Rightarrow C = -4$.

8. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + te^{-t}, \quad y(0) = y_0$$

remains finite as $t \longrightarrow \infty$

Solution:$\mu = e^{-t}$. y=-1-$\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t} + Ce^t$. $y(0) = y_0 \Rightarrow C = y_0 + \frac{5}{4}$.
The solution is y=-1-$\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t} + (y_0 + \frac{5}{4})e^t$. Since $e^t \to \infty$ as $t \to \infty$, we take y$_0$ = $\frac{-5}{4}$ so that the solution y=-1-$\frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$ is finite.