Do not need to hand in

Note that the final exam also covers material after HW8, including, for instance, calculating determinant by row operations, eigenvalues and eigenvectors, similarity and diagonalization etc. Although HW9 is not graded, you should work on it as serious as other homework assignments in order to be familiar with the above mentioned material.)

1. Find the determinant by reducing to triangular form for the following matrices.

(a) $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{bmatrix}$.

ANS: We perform the Gaussian Elimination on A by the following process:

$$\begin{aligned} A &= \begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{bmatrix} \\ &\rightarrow^{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 5 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{bmatrix} \\ &\rightarrow^{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & -6 \\ 0 & -1 & 2 \end{bmatrix} \\ &\rightarrow^{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 5 \\ 0 & -1 & 2 \\ 0 & 3 & -6 \end{bmatrix} \\ &\rightarrow^{R_3 \leftarrow R_3 + 3R_2} \begin{bmatrix} 1 & -1 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = U \end{aligned}$$

Note that matrix U is a triangular matrix and hence determinant of U is 0, the product of its diagonal elements.

Also note that interchange rows only effect the determinant by a sign (i.e., multiply by -1), and operation $R \leftarrow R - cR'$ does not change the determinant, and $R \leftarrow cR$ effect the determinant by a multiplication of c. Hence, $\det(A) = \det(U) = 0$.

(b) $A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 5 & 3 & 1 \\ -1 & 0 & 2 & -1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$.

ANS: We perform the Gaussian Elimination on A by the following process:

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 5 & 3 & 1 \\ -1 & 0 & 2 & -1 \\ 3 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1, R_4 \leftarrow R_4 - 3R_1}} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 7 & -1 & 5 \\ 0 & -1 & 4 & -3 \\ 0 & 4 & -4 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 7 & -1 & 5 \\ 0 & 4 & -4 & 6 \end{bmatrix}$$

$$\xrightarrow{\substack{R_4 \leftarrow R_4 + 4R_1 \\ R_3 \leftarrow R_3 + 7R_1}} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 27 & -16 \\ 0 & 0 & 12 & -6 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4/12} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 27 & -16 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 27 & -16 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - 27R_3} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -5/2 \end{bmatrix} = U$$

Note that matrix U is a triangular matrix and hence determinant of U is $5/2$, the product of its diagonal elements.

Also note that interchange rows only effect the determinant by a sign (i.e., multiply by -1), and operation $R \leftarrow R - cR'$ does not change the determinant, and $R \leftarrow cR$ effect the determinant by a multiplication of c. Hence, $\det(U) = \det(A) \cdot (-1)^2 \cdot 1/12 = 5/2$. Hence, $\det(A) = 30$.

2. Find the characteristic polynomial, the real eigenvalues and the corresponding eigenspaces of each of the following matrices:

(a): $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$.

ANS: Consider

$$A - \lambda I_2 = \begin{bmatrix} -\lambda - 1 & 3 \\ 2 & -\lambda \end{bmatrix},$$

with $\det(A - \lambda I_2) = \lambda(\lambda + 1) - 6 = \lambda^2 + \lambda - 6 = 0$. So the characteristic polynomial is $\lambda^2 + \lambda - 6$. Such an equation has two solution $\lambda_1 = 2$ and $\lambda_2 = -3$, which are eigenvalues of A.

For eigenvalue $\lambda_1 = 2$, one needs to find the eigenvectors for it, which is to solve the following homogeneous equation

$$(A - 2I_2)x = 0.$$

We can use Gauss-Elimination method to solve this as follows.

$$[A - 2I_2 \mid 0] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, there is one free variable $x_2 = t$ and $x_1 = x_2 = t$. So the solution is $x = \begin{bmatrix} t \\ t \end{bmatrix}$, and the eigenspace of A with respect to eigenvalue $\lambda_1 = 2$ is $\begin{bmatrix} t \\ t \end{bmatrix}$ for all real number t.

For eigenvalue $\lambda_2 = -3$, one needs to find the eigenvectors for it, which is to solve the following homogeneous equation

$$(A + 3I_2)x = 0.$$

We can use Gauss-Elimination method to solve this as follows.

$$[A + 3I_2 \mid 0] = \begin{bmatrix} 2 & 3 & \mid 0 \\ 2 & 3 & \mid 0 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & \mid 0 \\ 0 & 0 & \mid 0 \end{bmatrix}.$$

Hence, there is one free variable $x_2 = t$ and $x_1 = -3x_2/2 = -3t/2$. So the solution is $x = \begin{bmatrix} -3t/2 \\ t \end{bmatrix}$, and the eigenspace of A with respect to eigenvalue $\lambda_1 = -3$ is $\begin{bmatrix} -3t/2 \\ t \end{bmatrix}$ for all real number t.

(b): $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$.

ANS: First, let us calculate the determinant of $A - \lambda I_3$ as follows.

$$\begin{aligned} A - \lambda I_3 &= \begin{bmatrix} 1-\lambda & -2 & 3 \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{bmatrix} \\ &\to_{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1-\lambda \\ 2 & 6-\lambda & -6 \\ 1-\lambda & -2 & 3 \end{bmatrix} \\ &\to_{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + (\lambda-1)R_1}} \begin{bmatrix} 1 & 2 & -1-\lambda \\ 0 & 2-\lambda & 2\lambda-4 \\ 0 & 2\lambda-4 & 4-\lambda^2 \end{bmatrix} \\ &\to_{R_3 \leftarrow R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -1-\lambda \\ 0 & 2-\lambda & 2\lambda-4 \\ 0 & 0 & -(\lambda-2)^2 \end{bmatrix} = U. \end{aligned}$$

Hence, determinant of U is $(\lambda - 2)^3$.

Also note that interchange rows only effect the determinant by a sign (i.e., multiply by -1), and operation $R \leftarrow R - cR'$ does not change the determinant, and $R \leftarrow cR$ effect the determinant by a multiplication of c. Hence, $\det(U) = -\det(A - \lambda I_3)$ and then $\det(A - \lambda I_3) = -(\lambda - 2)^3$, which is the characteristic polynomial of A.

There is only one solution for $\det(A - \lambda I_3) = -(\lambda - 2)^3 = 0$, which is $\lambda = 2$. Then $\lambda = 2$ is the only eigenvalue of A.

Let us now calculate the eigenspace of A with respect to 2, that is all the solutions of $(A-2I_3)x=0$. We solve it by Gaussian Elimination method as follows.

$$[A-2I_3 \mid 0] = \left[\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \to_{\substack{R_3\leftarrow R_3+R_1 \\ R_2\leftarrow R_2+2R_1}} \left[\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This equation has two free variables $x_2 = s$ and $x_3 = t$, which leads $x_1 = 3t-2s$. Hence $x = \begin{bmatrix} 3t-2s \\ s \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ for all real numbers t, s is the eigenspace of A with respect to $\lambda = 2$.

3. Are the following matrices A and B similar to each other?

(a): $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$.

ANS: One can calculate the eigenvalues for A by solving $\det(A-\lambda I_2) = (\lambda-3)(\lambda+2) = 0$. That is $\lambda_1 = 3$ and $\lambda_2 = -2$ are the eigenvalues of A.

Similarly, one can calculate the eigenvalues for B by solving $\det(B-\lambda I_2) = (\lambda-6)(\lambda+1) = 0$. That is $\lambda_3 = 6$ and $\lambda_4 = -1$ are the eigenvalues of B.

Note that two similar matrices must have same eigenvalues. By the above information, A and B have different eigenvalues and hence they are not similar to each other.

(b): $A = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

ANS: One can calculate the eigenvalues for A by solving $\det(A-\lambda I_2) = (\lambda-5)(\lambda+1) = 0$. That is $\lambda_1 = 5$ and $\lambda_2 = -1$ are the eigenvalues of A.

Similarly, one can calculate the eigenvalues for B by solving $\det(B-\lambda I_2) = (\lambda-5)(\lambda-1) = 0$. That is $\lambda_3 = 5$ and $\lambda_4 = 1$ are the eigenvalues of B.

Note that two similar matrices must have same eigenvalues. By the above information, A and B have different eigenvalues and hence they are not similar to each other.

4. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

(a): $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$.

ANS: One can calculate the eigenvalues for A by solving $\det(A-\lambda I_2) = (\lambda-3)(\lambda+2) = 0$. That is $\lambda_1 = 3$ and $\lambda_2 = -2$ are the eigenvalues of A.

Now let us calculate the eigenvectors of A as follows.

For eigenvalue $\lambda_1 = 3$, one needs to find the eigenvectors for it, which is to solve the following homogeneous equation

$$(A-3I_2)x = 0.$$

We can use Gauss-Elimination method to solve this as follows.

$$[A - 3I_2 \mid 0] = \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, there is one free variable $x_2 = t$ and $x_1 = 4x_2 = 4t$. So the solution is $x = \begin{bmatrix} 4t \\ t \end{bmatrix}$. Let $t = 1$ and one gets a special eigenvector to be $u_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

For eigenvalue $\lambda_2 = -2$, one needs to find the eigenvalues for it, which is to solve the following homogeneous equation

$$(A + 2I_2)x = 0.$$

We can use Gauss-Elimination method to solve this as follows.

$$[A + 2I_2 \mid 0] = \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, there is one free variable $x_2 = t$ and $x_1 = -x_2 = -t$. So the solution is $x = \begin{bmatrix} -t \\ t \end{bmatrix}$. Take $t = 1$ and one gets a special eigenvector to be $u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Let matrix $P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$ with matrix $P^{-1} = 1/5 \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$. One can check that $P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, a diagonal matrix.

(b): $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

ANS: One can calculate the eigenvalues for A by solving $\det(A - \lambda I_2) = (\lambda - 4)(\lambda - 1) = 0$. That is $\lambda_1 = 4$ and $\lambda_2 = 1$ are the eigenvalues of A.

Now let us calculate the eigenvectors of A as follows.

For eigenvalue $\lambda_1 = 4$, one needs to find the eigenvectors for it, which is to solve the following homogeneous equation

$$(A - 4I_2)x = 0.$$

We can use Gauss-Elimination method to solve this as follows.

$$[A - 4I_2 \mid 0] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, there is one free variable $x_2 = t$ and $x_1 = x_2 = t$. So the solution is $x = \begin{bmatrix} t \\ t \end{bmatrix}$.

Let $t = 1$ and one gets a special eigenvector to be $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For eigenvalue $\lambda_2 = 1$, one needs to find the eigenvalues for it, which is to solve the following homogeneous equation

$$(A - I_2)x = 0.$$

We can use Gauss-Elimination method to solve this as follows.

$$[A - I_2 \mid 0] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, there is one free variable $x_2 = t$ and $x_1 = -2x_2 = -2t$. So the solution is $x = \begin{bmatrix} -2t \\ t \end{bmatrix}$. Take $t = 1$ and one gets a special eigenvector to be $u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Let matrix $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ with matrix $P^{-1} = 1/3 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. One can check that $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, a diagonal matrix.

5. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP = D$ is a diagonal matrix. Find D^{16}.

solution: $\det(A - \lambda I_2) = 0$ gives $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ as the eigenvalues of A.

When $\lambda_1 = 1 + i$, an eigenvector is $X_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

When $\lambda_1 = 1 - i$, an eigenvector is $X_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

Let matrix $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. It gives $P^{-1}AP = D$ with $D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$.

$D^{16} = \begin{bmatrix} (1+i)^{16} & 0 \\ 0 & (1-i)^{16} \end{bmatrix}$.

To simply, $(1 + i)^{16} = (\sqrt{2}e^{i\pi/4})^{16} = 2^8 e^{i4\pi} = 2^8$ and $(1 - i)^{16} = (\sqrt{2}e^{-i\pi/4})^{16} = 2^8 e^{-i4\pi} = 2^8$, since e^{it} is a periodic function with period 2π.