MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH2050 Assignment 8 Solutions

Due: Wednesday 22 November

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[10] 1. For each of the following matrices,

i. find the matrix M of minors, the matrix C of cofactors, the adjoint adj$A$, and compute $(\text{adj}A)A$ and $A(\text{adj}A)$.

ii. find det$A$.

iii. find A^{-1} or state why it doesn't exist.

(a) $A = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 5 & 2 \\ 4 & 8 & 9 \\ -1 & 2 & 5 \end{bmatrix}$

Solution.

(a)

i. $M = \begin{bmatrix} 14 & 8 \\ 7 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 14 & -8 \\ -7 & 4 \end{bmatrix}$ $\text{adj}A = \begin{bmatrix} 14 & -7 \\ -8 & 4 \end{bmatrix}$ $(\text{adj}A)A = A(\text{adj}A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

ii. $\det A = 4(14 - 7(8) = 0$.

iii. A does not have an inverse because it is singular ($\det A = 0$).

(b)

i. $M = \begin{bmatrix} 22 & 29 & 16 \\ 21 & 17 & 11 \\ 29 & 19 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 22 & -29 & 16 \\ -21 & 17 & -11 \\ 29 & -19 & 4 \end{bmatrix}$ $\text{adj}A = \begin{bmatrix} 22 & -21 & 29 \\ -29 & 17 & -19 \\ 16 & -11 & 4 \end{bmatrix}$

$(\text{adj}A)A = A(\text{adj}A) = \begin{bmatrix} -47 & 0 & 0 \\ 0 & -47 & 0 \\ 0 & 0 & -47 \end{bmatrix}$.

ii. $\det A = 3(22) + 5(-29) + 2(16) = -47$.

iii. $A^{-1} = \frac{1}{-47}\text{adj}A = \frac{-1}{47}\begin{bmatrix} 22 & -21 & 29 \\ -29 & 17 & -19 \\ 16 & -11 & 4 \end{bmatrix}$.

[6] 2. Use a Laplace expansion to find the determinant of each of the following matrices.

$A = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 1 \\ 3 & 5 & 2 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Solution.

The Laplace expansion along the second row of A yields

$$\begin{aligned} \det A &= 2\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} - \begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} \\ &= 2(5) - 11 \\ &= -1. \end{aligned}$$

The Laplace expansion along the fourth row of B yields

$$\begin{aligned}
\det B &= \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ -1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ -1 & 2 & 1 \end{vmatrix} \\
&= \left(- \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \right) - \left(\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \right) + \left(\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \right) \\
&= -(-3) - 3 - (-2) - 3(4) + 2 + 2(4) \\
&= 0.
\end{aligned}$$

You may have expanded along different rows at each step, but $\det B = 0$ regardless of your choice.

[6] 3. Are the following sets of vectors linearly independent or not?

(a) $\vec{a_1} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $\vec{a_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

(b) $\vec{b_1} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{b_3} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{b_4} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Solution.

(a) The vectors $\vec{a_1}$, $\vec{a_2}$ and $\vec{a_3}$ are linearly independent if and only if $\det A \neq 0$, where A is the 3×3 matrix whose columns are the vectors $\vec{a_1}$, $\vec{a_2}$ and $\vec{a_3}$. From question 2 we saw that $\det A = -1$, so $\vec{a_1}$, $\vec{a_2}$ and $\vec{a_3}$ are linearly independent.

Note: Since $\det A \neq 0$, what does that tell us about solutions to the equation $A\vec{x} = \vec{0}$?

(b) First notice that we can equivalently ask whether the vectors $\vec{b_3}$, $\vec{b_1}$, $\vec{b_2}$ and $\vec{b_4}$ are linearly independent (the order of the vectors doesn't matter). These vectors are linearly independent if and only if $\det B \neq 0$, where B is the matrix whose columns are given by the vectors $\vec{b_3}$, $\vec{b_1}$, $\vec{b_2}$ and $\vec{b_4}$. We saw in question 2 that $\det B = 0$, so $\vec{b_1}$, $\vec{b_2}$, $\vec{b_3}$ and $\vec{b_4}$ are linearly dependent.

[6] 4. Let A and B be 4×4 invertible matrices such that $\det A = -2$ and $\det B = 5$. Find the following determinants

(a) $\det A^{-1}$

(b) $\det \frac{1}{3} A$

(c) $\det A^{-1} B^T A^2 (2B)$.

Solution.

(a) $\det A^{-1} = \frac{1}{\det A} = -1/2$.

(b) $\det \frac{1}{3} A = \left(\frac{1}{3}\right)^4 \det A = -2/81$ since A is a 4×4 matrix.

(c)

$$\begin{aligned}
\det(A^{-1}B^TA^2(2B)) &= (\det A^{-1})(\det B^T)(\det A)(\det A)\det(2B) \quad \text{since the determinant is multiplicative} \\
&= \tfrac{1}{\det A}(\det B)(\det A)^2(2^4 \det B) \\
&= 2^4(\det A)(\det B)^2 \\
&= -(2^5)(25) = -800.
\end{aligned}$$

[4] 5. Suppose $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 5$. Find $\begin{vmatrix} 2p & 7a & a-x \\ 2q & 7b & b-y \\ 2r & 7c & c-z \end{vmatrix}$.

Solution.

$$\begin{aligned}
\begin{vmatrix} 2p & 7a & a-x \\ 2q & 7b & b-y \\ 2r & 7c & c-z \end{vmatrix} &= \begin{vmatrix} 2p & 2q & 2r \\ 7a & 7b & 7c \\ a-x & b-y & c-z \end{vmatrix} && \text{since } \det A^T = \det A \\
&= 14 \begin{vmatrix} p & q & r \\ a & b & c \\ a-x & b-y & c-z \end{vmatrix} && \text{by factoring 2 from the first row and 7 from the second row} \\
&= 14 \left(\begin{vmatrix} p & q & r \\ a & b & c \\ a & b & c \end{vmatrix} + \begin{vmatrix} p & q & r \\ a & b & c \\ -x & -y & -z \end{vmatrix} \right) && \text{by linearity of rows} \\
&= 14 \begin{vmatrix} p & q & r \\ a & b & c \\ -x & -y & -z \end{vmatrix} && \text{since } \det A = 0 \text{ if } A \text{ has two rows equal to each other} \\
&= -14 \begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} && \text{by factoring } -1 \text{ from the third row} \\
&= 14 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} && \text{by interchanging the first and second rows} \\
&= 70.
\end{aligned}$$

[8] 6. Without using the fact that $\det(AB) = (\det A)(\det B)$, show that $\det(EA) = (\det E)(\det A)$ for any $n \times n$ matrix A and elementary matrix E. (Consider separately the cases when A is singular and invertible).

Solution.

If $\det A = 0$ then A is not invertible, and neither is EA. Thus $\det(EA) = 0$ and $(\det E)(\det A) = 0$.

If A is invertible, then we consider the three different types of elementary matrices. If E is an elementary matrix corresponding to the row operation of interchanging two rows then $\det E = -1$ and $\det(EA) = -\det A$.

If E is an elementary matrix corresponding to the row operation of multiplying a row by a constant c, then $\det E = c$ (because E is a diagonal matrix with one diagonal entry equal to c and the remaining diagonal entries equal to 1) and $\det(EA) = c \det A$ (because we can factor c from the relevant row of A).

Finally, if E is an elementary matrix corresponding to the row operation of subtracting a multiple of one row from another, then $\det E = 1$ (because E is a triangular matrix whose diagonal entries are all equal to 1) and $\det(EA) = \det A$ (by linearity of the rows of a determinant).

Thus, in each case $\det(EA) = (\det E)(\det A)$.

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