MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 7 Solutions **Mathematics 2050** **fall 2017**

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[4] 1. Recall that a circle in the xy-plane has an equation of the form $x^2+y^2+ax+by+c=0$. Find the equation of the circle that passes through the points $(10,7)$, $(-4,-7)$ and $(-6,-1)$. Complete squares to find the centre and the radius of the circle.

Solution: Substituting the given points in the equation $x^2+y^2+ax+by+c=0$, we get the following system of linear equations with unknowns a, b and c:

$$\begin{cases} 10a+7b+c &= -149 \\ -4a-7b+c &= -65 \\ -6a-\ \ b+c &= -37 \end{cases}$$

Note that c occurs in each equation with coefficient 1, so Gaussian elimination will be easier if we treat c as the first variable: say, $x_1=c$, $x_2=b$, $x_3=a$. Then

$$\left[\begin{array}{ccc|c} 1 & 7 & 10 & -149 \\ 1 & -7 & -4 & -65 \\ 1 & -1 & -6 & -37 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & 10 & -149 \\ 0 & -14 & -14 & 84 \\ 0 & -8 & -16 & 112 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & 10 & -149 \\ 0 & 1 & 1 & -6 \\ 0 & 1 & 2 & -14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 7 & 10 & -149 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 1 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & 0 & -69 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -83 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -8 \end{array}\right]$$

Thus $a=x_3=-8$, $b=x_2=2$ and $c=x_1=-83$, so the equation of the circle is $x^2+y^2-8x+2y-83=0$. Completing the squares, we get $(x-4)^2-16+(y+1)^2-1-83=0$, so $(x-4)^2+(y+1)^2=100$. Thus the centre of the circle is $(4,-1)$ and the radius is 10.

[3] 2. Calculate AB and BA and determine whether A and B are inverses of each other.

(a) $A=\begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ -1 & -2 \\ 2 & 4 \end{bmatrix}$

(b) $A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $B=\frac{1}{24}\begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Solution:

(a) Neither A nor B has an inverse since neither is a square matrix. Indeed, we have

$$AB=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I, \text{ but } BA=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \neq I.$$

(This cannot happen with square matrices.)

(b) $AB=I=BA$, so A and B are inverses of one another.

[4] 3. Find all values of a for which the following homogeneous system has a nontrivial solution. Also, find all solutions.

$$\begin{cases} x - \ \ y - 2z &= 0 \\ x - 2y + az &= 0 \\ 2x + ay - 5z &= 0 \end{cases}$$

Solution: We apply Gaussian elimination to the coefficient matrix of the system (no need to carry an extra column of zeros):

$$\begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & a \\ 2 & a & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & a+2 \\ 0 & a+2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & a+2 \\ 0 & 0 & -1+(a+2)^2 \end{bmatrix}.$$

This is row echelon form. In order for the homogeneous system to have a nontrivial solution, we need a free variable. If $-1+(a+2)^2 = 0$, then z is a free variable; otherwise there are no free variables. So $(a+2)^2 = 1$, $a+2 = \pm 1$, which gives two values of a: $a = -3$ and $a = -1$.

If $a = -3$, the row echelon form is $\begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so $z = t$ is a free variable, $y = -z = -t$ and $x = y + 2z = t$. Hence $\mathbf{x} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

If $a = -1$, the row echelon form is $\begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $z = t$ is a free variable, $y = z = t$ and $x = y + 2z = 3t$. Hence $\mathbf{x} = t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

[5] 4. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$, find A^{-1} if it exists.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right].$$

Therefore, $A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

[3] 5. Given the matrices $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}$, find a matrix X such that $A^{-1}XA = B$.

Solution: $X = ABA^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 17 & 7 \end{bmatrix}$.

[4] 6. Given the matrix $C = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}$, find a matrix X such that $(X^T - 2I)^{-1} = C$.

Solution: Since $(X^T - 2I)^{-1} = C$, we have $X^T - 2I = C^{-1}$ and hence $X^T = C^{-1} + 2I$.

Therefore

$$X = \left(C^{-1} + 2I\right)^T = \left(\begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}^{-1} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \left(\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{13}{6} \end{bmatrix}$$

[3] 7. (a) If A and B are invertible $n \times n$ matrices that commute, prove that B and A^{-1} commute.

[4] (b) Let A be an $n \times n$ matrix such that $A^2 + 2A + I = 0$. Prove that A is invertible and find its inverse.

[4] (c) Let $A + I$ be invertible. Show that $(A + I)^{-1}$ and $(I - A)$ commute.

Solution:

(a) By hypothesis, we have $AB = BA$. Now multiplying both sides of this equation on both the right and the left by A^{-1} we obtain: $A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1} \Rightarrow BA^{-1} = A^{-1}B$.

(b) The equation satisfied by A can be rewritten as $-A^2 - 2A = I$. But then, factoring the left hand side yields $A(-A - 2I) = (-A - 2I)A = I$. From this it is clear that A is invertible with inverse $A^{-1} = -A - 2I$.

(c) We have:
$(A + I)(A - I) = A^2 - A + A - I = A^2 - I$
$(A - I)(A + I) = A^2 - A + A - I = A^2 - I$ So,
$(A + I)(A - I) = (A - I)(A + I)$
Multipling both sudes of this equation on both the right and the left by $(A + I)^{-1}$ gives the required result.

[6] 8. Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$.

(a) Express the matrix A as the product of elementary matrices.

(b) Express the matrix A^{-1} as the product of elementary matrices.

Solution:

(a) $$\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = E_1 A \to \begin{bmatrix} 1 & -2 \\ 0 & 7 \end{bmatrix} = E_2 E_1 A$$
$$\to \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = E_3 E_2 E_1 A \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = E_4 E_3 E_2 E_1 A$$

where

$$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{7} \end{bmatrix}, \qquad E_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Thus

$$A = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

(b) $$A^{-1} = E_4 E_3 E_2 E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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