MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 3 Solutions **Mathematics 2050** **fall 2017**

Due: October 4, 2017 . SHOW ALL WORK

[3] 1. Find the equation of a plane with x-intercept = 1, y-intercept = 2 and z-intercept = -3.

Solution: The scalar equation of a plane is given by $ax + by + cz = d$. The intercepts $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are all contained in the plane and hence satisfy the equation of the plane.

Therefore, x -int $= 1 \Rightarrow a = d$, y-int $= 2 \Rightarrow 2b = d$, and z int $= -3 \Rightarrow -3c = d$.

Hence, substituting these values into the equation of the plane we obtain:

$dx + \frac{d}{2}y - \frac{d}{3}z = d \Rightarrow x + \frac{y}{2} - \frac{z}{3} = 1 \Rightarrow 6x + 3y - 2z = 6$ is the equation of the required plane.

[3] 2. Find two vectors of length 3 which are perpendicular to both $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Solution: A vector perpendicular to both $\vec{u}$ and $\vec{v}$ is $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 5\vec{i} + \vec{j} - 3\vec{k}$.

$||\vec{u} \times \vec{v}|| = \sqrt{35}$ and so $\dfrac{1}{\sqrt{35}} \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$.

The vector $\dfrac{3}{\sqrt{35}} \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$ is a vector of length 3 perpendicular to both $\vec{u}$ and $\vec{v}$. Another one would be $-\dfrac{3}{\sqrt{35}} \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$.

[4] 3. Given $||\vec{u}|| = 10$, $||\vec{v}|| = 14$ and $||\vec{u} \times \vec{v}|| = 70$, find all possible values of $\vec{u} \cdot \vec{v}$.

Solution: $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin\theta \Rightarrow 70 = 10 \cdot 14 \sin\theta \Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$.

Now, $\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta$. Hence, there are two possibilities:

$\vec{u} \cdot \vec{v} = 10 \cdot 14 \cos\dfrac{\pi}{6}$ or $\vec{u} \cdot \vec{v} = 10 \cdot 14 \cos\dfrac{5\pi}{6} \Rightarrow \vec{u} \cdot \vec{v} = 70\sqrt{3}$ or $-70\sqrt{3}$.

4. Find the equation of the following planes.

[3] (a) the plane passing through the point (0, 1, 2) and containing the line $x = y = z$.

[3] (b) Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpen- dicular to the vector $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Solution: (a) Name Q(0, 1, 2). The line can be represented as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ which crosses the point $P(0, 0, 0)$ and is parallel to $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Set $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Now choose $\vec{n} = \vec{d} \times \overrightarrow{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \vec{i} - 2\vec{j} + \vec{k}$ and hence equation of the plane is $x - 2y + z = 0$ using the point $P(0, 0, 0)$.

(b) Choose $\vec{n} = \vec{u}$ since the two vectors are parallel. Hence, the equation of the required plane is $2x - y + 3z = d$. Since the plane passes through the point $P(1, 3, 5)$, $2(0) - 1 + 3(2) = d \Rightarrow d = 14$. Therefore, the equation of the plane is $2x - y + 3z = 14$.

5. Consider the points $A(1, -2, 1)$, $B(2, -2, -1)$ and $C(4, 1, 1)$.

[4] (a) Find the equation of the plane passing through A, B, and C.

[4] (b) Find the area of the triangle ABC.

Solution: (a) $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$.

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 3 & 3 & 0 \end{vmatrix} = 6\vec{i} - 6\vec{j} + 3\vec{k}$. Take $\vec{n} = 2\vec{i} - 2\vec{j} + \vec{k}$. Hence, equation of the required plane takes the form: $2x - 2y + z = d$. Using the point $A(1, -2, 1)$, we get $d = 7$. Therefore, $2x - 2y + z = 7$ is the equation of the requred plane.

(b)

$$\begin{aligned} \text{Area of } \Delta ABC &= \tfrac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| \\ &= \tfrac{1}{2}\left\|\begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}\right\| \\ &= \frac{9}{2} \end{aligned}$$

[4] 6. Find the intersection of the line $x = t, y = 2t, z = 3t$, and the plane $x + y + z = 1$.

Solution: Substituting $x = t$, $y = 2t$ and $z = 3t$ into the equation of the plane, we obtain: $t+2t+3t = 1 \Rightarrow 6t = 1 \Rightarrow t = \dfrac{1}{6}$ and the coordinates of the point of intdersection are $P(1/6, 1/3, 1/2)$.

[4] 7. Let ℓ_1 be the line with parametric equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \text{t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and ℓ_2 be the line described parametrically by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \text{s} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. How many planes are there which contain $\ell_2$and are parallel to ℓ_1 ? Find an equation describing one such plane.

Solution: Let $\vec{d_1}$ be the direction vector of the line ℓ_1 and let ℓ_2 be the direction vector of the line ℓ_2. Clearly, the lines ℓ_1 and ℓ_2 are not parallel since d_1 is not parallel to d_2. Hence, there is only one such plane and it must also be parallel to both $\vec{d_1}$ and $\vec{d_2}$. That is the plane is orthogonal to $\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 3 & 1 \end{vmatrix} = 3\vec{i} - \vec{j} + 3\vec{k}$. That is, $\vec{n} = 3\vec{i} - \vec{j} + 3\vec{k}$.
The equation of the required plane has the equation $3x - y + 3z = d$. Moreover, the plane passes through the point $P(2, 1, 0)$. Hence, $3(2) - 1 + 3(0) = d$. That is $d = 5$ and the equation of the plane is $3x - y + 3z = 5$.

[4] 8. Find the foot of the perpendicular from the point $P(1, -2, 3)$ to the plane $\pi : 3x+2y-z = 10$

Solution: Let Q be the foot of the perpendicular from $P(1, -2, 3)$ to the plane π : $3x + 2y - z = 10$. The line ℓ_{PQ} lies in the direction of the normal $\vec{n} = 3\vec{i} + 2\vec{j} - \vec{k}$ and passes through the point $P(1, -2, 3)$. Hence the equation of the line through PQ is given by:

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ and the parametric equations are $x = 1 + 3t$, $y = -2 + 2t$

and $z = 3 - t$. The intersection of this line with with the plane will be the point Q. The point on the line ℓ_{PQ} with parameter t lies on the plane $3x + 2y - z = 10$ if

$3(1 + 3t) + 2(-2 + 2t) - (3 - t) = 10 \Rightarrow t = 1$. Therefore, the coordinates of Q are:

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \text{ or } Q(4,0,2)$$

[4] 9. The line lying on the planes $x + y + z = 2$ and $3x - 4y + 5z = 6$

Solution: The direction vector of this line lies in both planes, so it is ortogonal to both of the normals $\vec{n_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{n_2} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$. Hence, $\vec{d} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{vmatrix} =$ $9\vec{i} - 2\vec{j} - 7\vec{k}$. Now we need to find a point on this line which is contained on both planes. We let $z = 0$ and solve for x and y. That is $x + y = 2 \,\text{and}\, 3x - 4y = 6 \Rightarrow x = 0, y = 0 \text{and} z = 0$. Therefore, $P(2,0,0)$ is a point on the line. Hence, equation of the required line is: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 9 \\ -2 \\ -7 \end{bmatrix}$