MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH2050 Assignment 4

Due: Wednesday 11 October

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[6] 1. Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be lines given by the following equations.

$$\ell_1\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \ell_2\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$

$$\ell_3\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \qquad \ell_4\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ -1 \end{bmatrix} + t_4 \begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix}$$

Determine whether the following pairs of lines are parallel, skew, or if they intersect. In each case explain your answer and find the intersection if it exists.

(a) ℓ_1 and ℓ_2. (b) ℓ_1 and ℓ_3. (c) ℓ_2 and ℓ_4.

Solution.

(a) Lines ℓ_1 and ℓ_2 are not parallel because there is no scalar c such that $d_1 = cd_2$. To determine whether they intersect, suppose there are scalars t_1 and t_2 so that

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$

Then this gives the following equations

$$\begin{aligned} 1 + 2t_1 &= -2 + 3t_2 \\ -1 + t_1 &= -3 + 2t_2 \\ t_1 &= 3 - 3t_2 \end{aligned}$$

Substituting the third of these equations into the second gives $-1 + 3 - t_2 = -3 + 2t_2$ which implies $t_2 = 1$ and $t_1 = 0$. This is also consistent with the first equations, therefore the lines ℓ_1 and ℓ_2 intersect at the point $(1, -1, 0)$.

(b) Lines ℓ_1 and ℓ_3 are not parallel because there is no scalar c such that $d_1 = cd_3$. To determine whether they intersect, suppose there are scalars t_1 and t_3 so that

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

Then this gives the following equations

$$\begin{aligned} 1 + 2t_1 &= 3 - 2t_3 \\ -1 + t_1 &= 2 + 4t_3 \\ t_1 &= 1 + t_3 \end{aligned}$$

Substituting the third of these equations into the second gives $-1 + 1 + t_3 = 2 + 4t_3$ which implies $t_3 = -\frac{2}{3}$, so then $t_1 = \frac{1}{3}$. However, from the first equation we have $1 + 2t_1 = \frac{5}{3}$ and $3 - 2t_3 = \frac{11}{3}$ and since $\frac{5}{3} \neq \frac{11}{3}$ the lines ℓ_1 and ℓ_3 do not intersect. Therefore ℓ_1 and ℓ_3 are skew.

(c) Lines ℓ_2 and ℓ_4 are parallel because their direction vectors are parallel. That is, $\begin{bmatrix} -6 \\ -4 \\ 6 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$. However note that they are not the same line because the point $(-2, -3, 3)$ is on ℓ_2 but not ℓ_4.

[6] 2. Let ℓ be the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$.

(a) Find the distance from $P(2, -1, 5)$ to ℓ.

(b) Find the point of ℓ closest to P.

Solution.

First note that $Q(-1, 1, 2)$ is a point on ℓ, then $\vec{QP} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$.

The (shortest) distance from P to ℓ is given by the length of the vector $\vec{QP} - \vec{p}$, where $\vec{p} = \text{proj}_{\ell}\vec{QP}$. So we first need to find $\vec{p}$. Note that $\vec{d} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ is the direction vector of ℓ.

$$\vec{p} = \frac{\vec{QP} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\vec{d} = \frac{-5}{26} \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

Then we have

$$\vec{QP} - \vec{p} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \frac{5}{26} \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 58/26 \\ -47/26 \\ 93/26 \end{bmatrix}$$

So the distance from P to ℓ is $\sqrt{\left(\frac{58}{26}\right)^2 + \left(\frac{47}{26}\right)^2 + \left(\frac{93}{26}\right)^2} = \sqrt{\frac{547}{26}}$.

(b) If $A(x, y, z)$ is the point of ℓ closest to P, then $\vec{AP} = \vec{QP} - \vec{p}$. So,

$$\begin{bmatrix} 2 - x \\ -1 - y \\ 5 - z \end{bmatrix} = \begin{bmatrix} 58/26 \\ -47/26 \\ 93/26 \end{bmatrix}$$

Thus, $2 - x = \frac{58}{26}$, $1 + y = \frac{47}{26}$ and $5 - z = \frac{93}{26}$. Therefore the point of ℓ closest to P is $\left(\frac{-6}{26}, \frac{73}{26}, \frac{37}{26}\right)$.

[6] 3. (a) Find two orthogonal vectors in the plane π given by the equation $2x - 3y + 4x = 0$.

(b) Given the vector $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find a vector $\vec{p}$ on the plane π such that $\vec{w} - \vec{p}$ is orthogonal to every vector on π.

Solution.

(a) To find two orthogonal vectors $\vec{e}$ and $\vec{f}$ in π we start with two nonparallel vectors u and v in the plane. The vectors $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ are one such pair (you may have found a different pair).

Now find the projection $\vec{r} = \text{proj}_{\vec{v}}\vec{u}$.

$$\vec{r} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{6}{13} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

Let $\vec{e} = \vec{v}$ and $\vec{f} = \vec{u} - \vec{r} = \begin{bmatrix} 8/13 \\ -12/13 \\ -1 \end{bmatrix}$. Then the vectors $\vec{e}$ and $\vec{f}$ are the required orthogonal vectors in π.

(b) Find the projection $\vec{p} = \text{proj}_{\pi}\vec{w}$.

Use the orthogonal vectors $\vec{e}$ and $\vec{f}$ from part (a).

$$\begin{aligned}\text{proj}_{\pi}\vec{w} &= \tfrac{\vec{w}\cdot\vec{e}}{\vec{e}\cdot\vec{e}}\vec{e} + \tfrac{\vec{w}\cdot\vec{f}}{\vec{f}\cdot\vec{f}}\vec{f} \\ &= \tfrac{5}{13}\begin{bmatrix}3\\2\\0\end{bmatrix} + \tfrac{-17/13}{(8/13)^2+(12/13)^2+(13/13)^2}\begin{bmatrix}8/13\\-12/13\\-1\end{bmatrix} = \tfrac{5}{13}\begin{bmatrix}3\\2\\0\end{bmatrix} - \tfrac{17}{29}\begin{bmatrix}8/13\\-12/13\\-1\end{bmatrix} \\ &= \begin{bmatrix}23/29\\38/29\\17/29\end{bmatrix}.\end{aligned}$$

[4] 4. Let $\vec{u} = \begin{bmatrix}2\\4\\3\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$. Find $\text{proj}_{\vec{u}}\vec{v}$ and $\text{proj}_{\vec{v}}\vec{u}$.

Solution.

The projection of $\vec{v}$ on the vector $\vec{u}$ is

$$\begin{aligned}\text{proj}_{\vec{u}}\vec{v} &= \tfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\ &= \tfrac{4}{29}\begin{bmatrix}2\\4\\3\end{bmatrix} = \begin{bmatrix}8/29\\16/29\\12/29\end{bmatrix}.\end{aligned}$$

The projection of $\vec{u}$ on the vector $\vec{v}$ is

$$\begin{aligned}\text{proj}_{\vec{v}}\vec{u} &= \tfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \\ &= \tfrac{4}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}2/3\\-2/3\\4/3\end{bmatrix}.\end{aligned}$$

[6] 5. Given $P(1,1,3)$ and a plane π with equation $x + 3y - z = 7$.

(a) Find the distance from point P to the plane π.

(b) Find the point on π closest to P.

Solution.

(a) The (shortest) distance from point P to the plane π is given by the length of the vector $\text{proj}_{\vec{n}}\vec{PQ}$, where $\vec{n}$ is a normal vector to the plane and Q is a point on the plane.

The point $Q(1,2,0)$ is on the plane π (you may have found a different point) so let $\vec{PQ} = \begin{bmatrix}0\\1\\-3\end{bmatrix}$, and $\vec{n} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$ is normal to π.

$$\text{proj}_{\vec{n}}\vec{PQ} = \frac{\vec{PQ}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{6}{11}\vec{n}$$

Then the shortest distance from P to π is given by $\|\frac{6}{11}\vec{n}\| = \frac{6}{11}\|\vec{n}\| = 6\sqrt{11}/11$.

(b) The point on π closest to P is the point $A(x,y,z)$ with $\vec{PA} = \text{proj}_{\vec{n}}\vec{PQ}$. So we have

$$\begin{bmatrix}x-1\\y-1\\z-3\end{bmatrix} = \frac{6}{11}\begin{bmatrix}1\\3\\-1\end{bmatrix}$$

Therefore $x - 1 = \frac{6}{11}$, $y - 1 = \frac{18}{11}$ and $z - 3 = -\frac{6}{11}$. So $A\left(\frac{17}{11}, \frac{29}{11}, \frac{27}{11}\right)$ is the point on π closest to P.

[4] 6. Describe in geometrical terms the set of all points one unit away from the plane with equation $2x - 2y + z = 3$. Find an equation for this set of points.

Solution. Let π be the plane given by the equation $2x - 2y + z = 3$. The collection of points with (shortest) distance 1 to the plane π will form two planes parallel to π, one on either "side" of π.

To find the equations for these planes we consider points $P(x, y, z)$ at distance 1 from π. The distance from P to π is given by the length of the vector $\text{proj}_{\vec{n}}\vec{PQ}$, where Q is a point on π and $\vec{n}$ is normal to π.

Let $Q(1, 0, 1)$, then $\vec{PQ} = \begin{bmatrix} 1-x \\ -y \\ 1-z \end{bmatrix}$ and note that $\vec{n} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

$$\text{proj}_{\vec{n}}\vec{PQ} = \frac{\vec{PQ} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}\vec{n} = \frac{2(1-x) + 2y + (1-z)}{9}\vec{n}$$

So because $P(x, y, z)$ is one unit away from π we have

$$1 = \left\| \frac{2(1-x) + 2y + (1-z)}{9}\vec{n} \right\| = \left| \frac{3 - 2x + 2y - z}{9} \right| \|\vec{n}\| = |3 - 2x + 2y - z| \, \tfrac{3}{9}$$

Rearranging this and considering both cases for the absolute value gives the equations

$$3 = \pm(3 - 2x + 2y - z)$$

This simplifies to the following equations for the two planes one unit away from π

$$2x - 2y + z = 0 \quad \text{and} \quad 2x - 2y + z = 6.$$

[2] 7. Suppose vectors $\vec{u}$ and $\vec{v}$ are linearly independent. Show that the vectors $\vec{w} = \vec{u} - \vec{v}$ and $\vec{z} = \vec{u} + 3\vec{v}$ are also linearly independent.

Solution.

Suppose $c\vec{w} + d\vec{z} = \vec{0}$, then this implies

$$\begin{aligned} \vec{0} &= c(\vec{u} - \vec{v}) + d(\vec{u} + 3\vec{v}) & \\ &= c\vec{u} - c\vec{v} + d\vec{u} + 3d\vec{v} & \text{by distributivity} \\ &= (c + d)\vec{u} + (3d - c)\vec{v} & \text{again, by distributivity} \end{aligned}$$

It follows that $c + d = 0$ and $3d - c = 0$ because $\vec{u}$ and $\vec{v}$ are linearly independent. Therefore $c = -d$ and $c = 3d$, so the only solution is that $c = d = 0$ which shows that $\vec{w}$ and $\vec{z}$ are linearly independent.

[6] 8. Determine whether or not the following sets of vectors are linearly independent or linearly dependent. If the vectors are linearly dependent then give an example of a nontrivial linear combination of the vectors which equals the zero vector.

(a) $\vec{u} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$ (b) $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$ (c) $\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{0}$

Solution.

(a) Suppose $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ for scalars a, b and c. Then we have the following equations.

$$\begin{aligned} 2a + b + 5c &= 0 \\ 4a - b + 6c &= 0 \\ 3a + 9c &= 0 \end{aligned}$$

The third of these equations implies $a = -3c$, which if we substitute into the second equation yields $-12c - b + 6c = 0$ so $b = -6c$. Finally, by considering the first equation $-6c - 6c + 5c = -7c = 0$. Therefore the only solution to the above equations is $a = b = c = 0$, so $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly independent.

(b) Suppose $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ for scalars a, b and c. Then we have the following equations.

$$\begin{aligned}2a - b + 7c &= 0\\ 3b - 3c &= 0\\ a + 2b + c &= 0\end{aligned}$$

The second of these equations implies $c = b$ which we can substitute into the third equation to obtain $a + 3b = 0$, that is $a = -3b$. By substituting this into the first equation we have $-6b - b + 7b = 0$ which holds for any value b. So an example solution is $b = 1$, $c = 1$ and $a = -3$.

(c) These vectors are linearly dependent. Nontrivial solutions are $0\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 + c\vec{0} = \vec{0}$, where c is any nonzero scalar.

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