Solution.

MEMORIAL UNIVERSITY

DEPARTMENT OF MATHEMATICS & STATISTICS

TEST 2 | **Math 2050** | **FALL 2017**

Last Name: **First name:** **Student ID:**

[8] 1. Assume $\vec{u} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $\vec{w} = \begin{bmatrix}0\\1\\1\end{bmatrix}$. Is vector $\vec{x} = \begin{bmatrix}-1\\2\\1\end{bmatrix}$ a linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$?

Assume $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{x}$. This gives $\begin{bmatrix}1 & 1 & 0\\-1 & 1 & 1\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-1\\2\\1\end{bmatrix}$.

$$[A|b] = \left[\begin{array}{ccc|c}1 & 1 & 0 & -1\\-1 & 1 & 1 & 2\\1 & 2 & 1 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1 & 1 & 0 & -1\\0 & 2 & 1 & 1\\0 & 1 & 1 & 2\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 1 & 0 & -1\\0 & 1 & 1 & 2\\0 & 2 & 1 & 1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c}1 & 1 & 0 & -1\\0 & 1 & 1 & 2\\0 & 0 & -1 & -3\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 1 & 0 & -1\\0 & 1 & 1 & 2\\0 & 0 & 1 & 3\end{array}\right]$$

$c_1 = 0$, $c_2 = -1$, $c_3 = 3$.

$\vec{x}$ is a linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$.

[9] 2. Find conditions on a and b such that the system

$$\begin{cases} -x + 3y + 2z = -8 \\ x + z = 2 \\ 3x + 3y + az = b \end{cases}$$

has a

(a) unique solution;

(b) no solution;

(c) infinitely many solutions.

$$\left[\begin{array}{ccc|c}-1 & 3 & 2 & -8\\1 & 0 & 1 & 2\\3 & 3 & a & b\end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + 3R_1]{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{ccc|c}-1 & 3 & 2 & -8\\0 & 3 & 3 & -6\\0 & 12 & 6+a & b-24\end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 4R_2} \left[\begin{array}{ccc|c}-1 & 3 & 2 & -8\\0 & 3 & 3 & -6\\0 & 0 & a-6 & b\end{array}\right]$$

Therefore. (1). If $a \neq 6$, there is a unique solution.

(2) If $a = 6$, $b \neq 0$, there is no solution

(3) If $a = 6$ and $b = 0$, there are infinitely many solutio[ns]

[10] 3. Solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - 2x_4 = 0 \\ -3x_1 + 6x_2 + x_3 = 4 \\ -2x_1 + 4x_2 + 4x_3 - 2x_4 = 4 \end{cases}$$

Express the solution of in the form $X = X_p + X_h$ where X_p is a particular solution and X_h is a solution to the corresponding homogeneous system.

$$[A|b] = \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 4 \\ -2 & 4 & 4 & -2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 0 & 0 & 10 & -6 & 4 \\ 0 & 0 & 10 & -6 & 4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 0 & 0 & 10 & -6 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

let $x_2 = t$, $x_4 = s$. $\quad 10x_3 = 4 + 6x_4 = 4 + 6s$

$$x_3 = \frac{2}{5} + \frac{3}{5}s$$

$$x_1 = 2x_2 - 3x_3 + 2x_4 = 2t - 3\left(\frac{2}{5} + \frac{3}{5}s\right) + 2s$$
$$= -\frac{6}{5} + 2t + \frac{1}{5}s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6/5 \\ 0 \\ 2/5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1/5 \\ 0 \\ 3/5 \\ 1 \end{bmatrix}.$$

$$X_p = \begin{bmatrix} -6/5 \\ 0 \\ 2/5 \\ 0 \end{bmatrix}, \quad X_h = t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1/5 \\ 0 \\ 3/5 \\ 1 \end{bmatrix}.$$

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[8] 5. (a) Find the inverse of $A = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 6 \\ 1 & 0 & 1 \end{bmatrix}$

[3] (b) Solve the matrix equation $AX = b$ where A is the matrix of part (a) and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(You may use A^{-1} from part (a).)

(a) $$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 1 & 4 & 1 & 0 & 0 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 3 & 1 & 4 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3 \to R_3]{-4R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -4 \\ 0 & 1 & 1 & 1 & 0 & -3 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -4 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{(-1)R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & -2 \\ -1 & 1 & -1 \end{bmatrix}$$

(b). $$X = A^{-1}b = \begin{bmatrix} 5 \\ -6 \\ -2 \end{bmatrix}$$

[6] 5. (a) Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Find the matrix X if $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} XA = \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}$

$$XA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} A^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}{2-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}}{0-2} = -\frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

[6] (b) Express the invertible matrix A as the product of elementary matrices.

$$A \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{r_2 \to \frac{1}{2} r_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 + (-1) r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$E_3 E_2 E_1 A = I$$

$$A = (E_3 E_2 E_1)^{-1}$$

$$= E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

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