MEMORIAL UNIVERSITY

DEPARTMENT OF MATH & STAT

Solution

TEST 1 | Math 2050 | [redacted]

Last Name: First name: **Student ID:**

[9] 1. Let $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix}$.

(1) Find $||\vec{u}||$ and $||\vec{v}||$.

(2) Find the angle between $\vec{u}$ and $\vec{v}$.

(3) Find a vector of length 5 in the opposite direction of $\vec{u}$.

(1) $||\vec{u}|| = \sqrt{1^2+3^2+1^2} = \sqrt{11}$, $||\vec{v}|| = \sqrt{(-2)^2+4^2+(-10)^2} = \sqrt{120}$

(2) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||} = \dfrac{1\times(-2)+3\times4+1\times(-10)}{\sqrt{11}\ \sqrt{120}} = 0$

$\theta = \frac{\pi}{2}$.

(3) unit vector of $\vec{u}$ is $\frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$-5\cdot\frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-5}{\sqrt{11}} \\ -15/\sqrt{11} \\ -5/\sqrt{11} \end{bmatrix}$ is the answer.

[5] 2. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors of length 3,2 and 6 respectively. Suppose $\vec{u}\cdot\vec{v} = 3$, $\vec{u}\cdot\vec{w} = 1$ and $\vec{v}\cdot\vec{w} = 2$. Find $(3\vec{u} - 2\vec{v})\cdot(5\vec{v} - \vec{w})$.

$(3\vec{u} - 2\vec{v})\cdot(5\vec{v} - \vec{w})$

$= 15\vec{u}\cdot\vec{v} - 3\vec{u}\cdot w - 10\vec{v}\cdot v + 2\vec{v}\cdot\vec{w}$

$= 15\times3 - 3\cdot1 - 10\cdot2^2 + 2\cdot2$

$= 6$

[10] 3. (a) Find the equation of the plane which contains the points $A(1,2,3)$, $B(1,5,2)$ and $C(3,-1,1)$.

(b) Find the area of the triangle ABC.

(a) $\vec{AB} = \begin{bmatrix} 1-1 \\ 5-2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$, $\vec{AC} = \begin{bmatrix} 3-1 \\ -1-2 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$

$\vec{AB} \times \vec{AC} = \begin{bmatrix} i & j & k \\ 0 & 3 & -1 \\ 2 & -3 & -2 \end{bmatrix} = \begin{bmatrix} \cancel{8}\,-9 \\ -2 \\ -6 \end{bmatrix}$ is the normal vector.

The plane equation is

$-9(x-1) - 2(y-2) - 6(z-3) = 0$

$-9x - 2y - 6z = \cancel{-9}\ -31$

(b) $S_{\triangle ABC} = \frac{1}{2} \|\vec{AB} \times \vec{AC}\| = \frac{1}{2} \left\| \begin{bmatrix} -9 \\ -2 \\ -6 \end{bmatrix} \right\| = \frac{1}{2}\sqrt{(-9)^2 + (-2)^2 + 6^2}$

$= \frac{1}{2}\sqrt{121} = \frac{11}{2}$

[6] 4. Does the line l_1 $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ intersect the line l_2 $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$?

If yes, find the point of intersection.

Rewrite the line l_2 as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$. This equation describes the same points as those in l_2.

Let $\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$

i.e. $\begin{cases} 2 + t = 1 + 4s \\ -3 + 2t = 2 - 6s \\ -4 + 3t = 0 - 2s \end{cases} \Rightarrow t = 1,\ s = \frac{1}{2}$

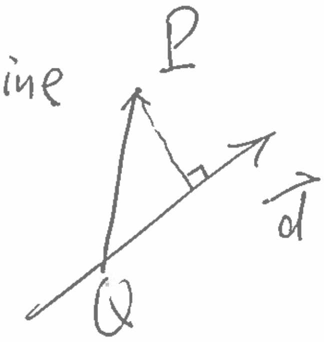
the intersection point is $(3, -1, -1)$.

Two lines do intersect.

[14] 5. (a) Find the distance from $P(3,1,-1)$ to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

(b) Find the distance from $P(3,1,-1)$ to the plane $3x - 4y + z = 18$.

(a). Find a point $Q\,(1,0,2)$ on the line



$$\overrightarrow{QP} = \begin{bmatrix} 3-1 \\ 1-0 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}. \quad \vec{d} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

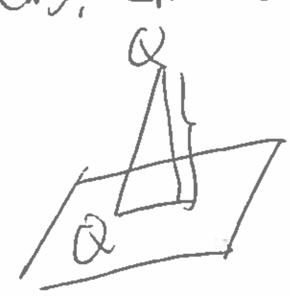
$$\operatorname{proj}_{\vec{d}} \overrightarrow{QP} = \frac{\overrightarrow{QP}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{-1}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The distance $= \|\overrightarrow{QP} - \operatorname{proj}_{\vec{d}}\overrightarrow{QP}\| = \left\| \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} - \frac{(-1)}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\|$

$$= \left\| \frac{1}{5}\begin{bmatrix} 10 \\ 7 \\ -14 \end{bmatrix} \right\| = \frac{1}{5}\sqrt{10^2+7^2+14^2}$$

$$= \frac{1}{5}\sqrt{345}$$

(b). choose $Q\,(3,0,9)$ on the plane.



$$\overrightarrow{QP} = \begin{bmatrix} 3-3 \\ 1-0 \\ -1-9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -10 \end{bmatrix}, \quad \vec{n} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\operatorname{proj}_{\vec{n}} \overrightarrow{QP} = \frac{\overrightarrow{QP}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{-14}{3^2+(-4)^2+1^2}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

The distance $= \|\operatorname{proj}_{\vec{n}}\overrightarrow{QP}\| = \left\| \frac{-14}{26}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\|$

$$= \frac{14}{\sqrt{26}} = \frac{7}{13}\sqrt{26}.$$

[8] 6. Are vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ linearly independent?

[50]

Suppose $k_1\vec{u} + k_2\vec{v} + k_3\vec{w} + k_4\vec{x} = \vec{0}$

eg:
$$\begin{cases} k_1 + k_2 + k_3 - k_4 = 0 \\ -k_1 + k_2 \quad + 2k_4 = 0 \\ k_1 \quad + k_3 \quad = 0 \end{cases}$$

from the last equation $k_1 = -k_3$, substituting it into the first and the second:

$$\begin{cases} k_2 - k_4 = 0 \Rightarrow k_2 = k_4 \\ k_2 + k_3 + 2k_4 = 0 \Rightarrow 3k_4 + k_3 = 0 \\ \qquad \Rightarrow k_3 = -3k_4. \end{cases}$$

Finally we have

$$\begin{cases} k_1 = 3k_4 \\ k_2 = k_4 \\ k_3 = -3k_4 \end{cases} \quad \text{for free } k_4$$

when $k_4 = 0$, $\Rightarrow k_1 = k_2 = k_3 = 0$.

when $k_4 = 1$, $\Rightarrow (k_1, k_2, k_3, k_4) = (3, 1, -3, 1)$

The system has at least one non-zero solution $(3, 1, -3, 1)$. Therefore, the vectors are linearly dependent.