Math 3202 **Solutions** **Assignment #8(extra credit)**

1. Is the following vector field irrotational or incompressible at point $(0, 1, 2)$?

$$\vec{F} = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right)$$

Solution:

1. From Assignment 7 we know that curl $\vec{F} = \vec{0} = (0, 0, 0)$. Thus the field is irrotational at point $(0, 1, 2)$. In fact it is irrotational at any point but the origin where it is not defined.

2. Compute div$\vec{F} = (x^2 + y^2 + z^2)^{-1}$ it gives $1/5$ at point $(0, 1, 2)$. Since it is not zero, the field is not incompressible at this point.

2. Evaluate the surface integral for given function

(a) $\int\int_{\mathbf{S}} y\, dS$, where $\mathbf{S}$ is a surface $z = 2/3(x^{3/2} + y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$.
Solution: The normal vector to the surface is $\vec{n} = (x^{1/2}, y^{1/2}, -1)$. Thus

$$\int\int_{\mathbf{S}} y\, dS = \int_0^1 \int_0^1 y(x + y + 1)^{1/2}\, dxdy = \frac{4}{105}(9\sqrt{3} + 4\sqrt{2} - 2).$$

Here to evaluate the y-integral it is convenient to sub $u = y + 2$ or $u = y + 1$.

(b) $\int\int_{\mathbf{S}} \sqrt{1 + x^2 + y^2}\, dS$, where $\mathbf{S}$ is the helicoid with vector equation $\vec{r}(u, v) = (u \cos v, u \sin v, v)$, $0 \le u \le 1$, $0 \le v \le \pi$.
Solution: The normal vector to the surface is $\vec{n} = \vec{r}_u \times \vec{r}_v = (\sin v, -\cos v, u)$. Its length is $(1 + u^2)^{1/2}$. Thus

$$\int\int_{\mathbf{S}} \sqrt{1 + x^2 + y^2}\, dS = \int_0^\pi \int_0^1 (1 + u^2)^{1/2}(1 + u^2)^{1/2}\, dudv = 4\pi/3.$$

3. Evaluate the surface integral for given vector field

(a) $\int\int_{\mathbf{S}} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = (xy, 4x^2, yz)$ and $\mathbf{S}$ is a surface $z = xe^y$, $0 \le x \le 1$, $0 \le y \le 1$, with upward orientation.
Solution: The normal upward vector to the surface is $\vec{n} = (-e^y, -xe^y, 1)$. Thus

$$\int\int_{\mathbf{S}} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 -xy(e^y) - 4x^2(xe^y) + y(xe^y) dxdy = 1 - e.$$

(b) $\int\int_{\mathbf{S}} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = (y, x, z^2)$ and $\mathbf{S}$ is the helocoid with vector equation $\vec{r}(u, v) = (u \cos v, u \sin v, v)$, $0 \le u \le 1$, $0 \le v \le \pi$, with upward orientation.
Solution: The normal upward vector to the surface is $\vec{n} = (\sin v, -cosv, u)$, and $\vec{F}(\vec{r}(u, v)) = (u \sin v, u \cos v, v^2)$. Thus

$$\int\int_{\mathbf{S}} \vec{F} \cdot d\vec{S} = \int_0^\pi \int_0^1 (u \sin^2 v - u \cos^2 v + uv^2)\, dudv = \pi^3/6$$

4. Use Stokes's Theorem to evaluate surface integral $\int\int_{\mathbf{S}} \text{curl}\vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = (yz, xz, xy)$, and surface $\mathbf{S}$ is a part of paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

   Solution: The plane $z = 5$ intersects the paraboloid in the circle $z = 5, x^2 + y^2 = 4$. Perametric equation of the circle is $x = 2\cos t, y = 2\sin t, z = 5,\ 0 \le t \le 2\pi$. The vector field on the curve is $\vec{F} = (10\sin t, 10\cos t, 4\cos t\sin t)$.

   By Stokes' Theorem

$$\int\int_{\mathbf{S}} \text{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-20\sin^2 t + 20\cos^2 t)\, dt = 0.$$

5. Use Stokes's Theorem to evaluate line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (e^{-x}, e^x, e^z)$, and C is the boundary of the plane $2x + y + 2z = 2$ in the first octant, oriented counterclockwise as viewed from above.

   Solution: Here $\text{curl}\vec{F} = (0, 0, e^x)$. The surface is the portion of the plane $z = (2 - 2x - y)/2$ over triangular region in xy-plane $0 \le x \le 1,\ 0 \le y \le 2 - 2x$.

   By Stokes' Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_{\mathbf{S}} \text{curl}\vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2-2x} e^x dy dx = 2e - 4.$$

6. Verify that Divergence theorem is true for the vector field $\vec{F} = (x^2, xy, z)$ and the solid bounded by paraboloid $z = 4 - x^2 - y^2$ and xy-plane.

   Solution:

   1. $\text{div}\vec{F} = 3x + 1 = 3r\cos\theta + 1$. Thus

$$\int\int\int_E \text{div}\vec{F} dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r\cos\theta + 1) r dz dr d\theta = 8\pi$$

   2a Surfase integral through the top (paraboloid) is 8π.

   Here normal outward vector is $\vec{n} = (2x, 2y, 1)$ and vector field on the surface is $\vec{F} = (x^2, xy, 4 - x^2 - y^2)$. The domain of integration is the circle $x^2 + y^2 \le 4$ in the xy-plane.

   2b Surface integral through the bottom (circle in xy-plane) is zero.

   Here normal outward vector is $\vec{n} = (0, 0, -1)$ and vector field on the surface is $\vec{F} = (x^2, xy, 0)$. The domain of integration is the circle $x^2 + y^2 \le 4$ in the xy-plane.

   Thus total flux is the same as the triple integral.

7. Verify that Divergence theorem is true for the vector field $\vec{F} = (x, y, z)$ and the unit ball $x^2 + y^2 + z^2 = 1$.

   Solution:

   1. div$\vec{F} = 3$. Thus the triple integral is $\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\, d\rho d\phi d\theta = 4\pi$.
   2. The surface integral is

   $$\int_0^{2\pi} \int_0^{\pi} (\sin^3 v \cos^2 u + \sin^3 v \sin^2 u + \sin v \cos^2 v)\, dv du = 4\pi.$$

   Here we used parametric equation for the sphere

   $$x = \sin v \cos u, \quad y = \sin v \sin u, \quad z = \cos v, \qquad 0 \le v \le \pi, \quad 0 \le u \le 2\pi$$

   with outward normal vector

   $$\vec{n} = (\sin^2 v \cos u,\ \sin^2 v \sin u,\ \sin v \cos v).$$