Math 3202 **Solutions** **Assignment #5**

1. Find volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

   Solution: In cylindrical coordinates the volume is bounded by cylinder $r=1$ and sphere $r^2+z^2=4$. Thus the integral is $\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dzdrd\theta = (4/3)\pi(8-3\sqrt{3})$.

2. Find the mass of a ball of radius a if the density at any point is proportional to its distance from a fixed axis of symmetry of the ball.

   Solution: Let the axis of the symmetry be z. Then the (shortest) distance from any point (x,y,z) to the axis is $\sqrt{x^2+y^2}$. Thus the density is $k\sqrt{x^2+y^2} = kr = k\rho\sin\phi$, where k is a constant. Then the mass of the ball in spherical coordinates is $\int_0^{2\pi}\int_0^{\pi}\int_0^{a} k\rho\sin\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta = a^4\pi^2k/4$

3. Evaluate $\int\int\int_H (x^2+y^2)dV$, where H is the semispherical region below $x^2+y^2+z^2=1$ and above xy-plane.

   Solution: In spherical coordinates it becomes $\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\sin^2\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta = 4\pi/15$

4. Evaluate $\int\int\int_H e^{\sqrt{x^2+y^2+z^2}}dV$, where H is enclosed by sphere $x^2+y^2+z^2=9$ in the first octant.

   Solution: In spherical coordinates it becomes $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\rho}\,\rho^2\sin\phi\,d\rho d\phi d\theta = (\pi/2)(5e^3-2)$.

5. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane and below the cone $z=\sqrt{x^2+y^2}$.

   Solution: In sperical coordinates this solid is $0\le\theta\le 2\pi$, $\pi/4\le\phi\le\pi/2$, $0\le\rho\le 2$ Thus the volume is

   $\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\phi\,d\rho d\phi d\theta = 8\sqrt{2}\pi/3$.

6. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

   Solution: In sperical coordinates this solid is $0\le\theta\le\pi/6$, $0\le\phi\le\pi$, $0\le\rho\le a$ Thus the volume is

   $\int_0^{\pi}\int_0^{\pi/6}\int_0^a \rho^2\sin\phi\,d\rho d\phi d\theta = \pi a^3/9$.

   P.S. Note that this *orange piece* is (1/12) portion of the sphere thus the answer is $((4/3)\pi a^3)/12 = \pi a^3/9$.

7. Evaluate using spherical coordinates.

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)dz\,dx\,dy.$$

   Solution: First we identify the solid as being a quater of an *ice-cream cone* bounded by cone $z=r$ and sphere $z=\sqrt{18-r^2}$. The line of intersection of the cone and the sphere is found from $r=\sqrt{18-r^2}$, thus $z=r=3$. In the xy-plane we have a quarter of a circle $0\le\theta\le\pi/2$, $0\le r\le 3$. Also, $r\le z\le\sqrt{18-r^2}$.

   This solid in spherical coordinates is $0\le\rho\le\sqrt{18}$, $0\le\phi\le\pi/4$, $0\le\theta\le\pi/2$. The integrant is ρ^2 thus the integral becomes

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)dz\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{18}}\rho^4\sin\phi\,d\rho d\phi d\theta = 486\pi\frac{\sqrt{2}-1}{5}.$$

8. Use transformation $x = \sqrt{2}\,u - \sqrt{2/3}\,v$, $y = \sqrt{2}\,u + \sqrt{2/3}\,v$ to evaluate $\int\int_R (x^2 - xy + y^2)dA$, where the region R is bounded by $x^2 - xy + y^2 = 2$.

   Solution: The Jacobian of the change of variables is $4/\sqrt{3}$. $x^2 - xy + y^2 = 2u^2 + 2v^2$. Thus in new variables the domain of integration is the circle $u^2 + v^2 \leq 1$. Another change of variables $u = r\cos\theta$, $u = r\sin\theta$ leads to

   $\int\int_R (x^2 - xy + y^2)dA = \frac{4}{\sqrt{3}} \int_0^1 \int_0^{2\pi} 2r^2\, r d\theta dr = 4\pi/\sqrt{3}$.