Math 3202 **Answers** **Assignment #2**

1. Let trajectory be given by vector function $\vec{r}(t) = (t\cos(2t), t\sin(2t), 0)$. Find velocity vector $\vec{v}(t) = \vec{r}'$, angular momentum vector $\vec{L}(t) = \vec{r} \times \vec{v}$, acceleration vector $\vec{a}(t) = \vec{r}''$ and torque vector $\vec{\tau}(t) = \vec{r} \times \vec{a}$. Is there a moment of time where vectors $\vec{r}$ and $\vec{v}$ are perpendicular?

   Answers $\vec{v} = (\cos(2t) - 2t\sin(2t), \sin(2t) + 2t\cos(2t), 0)$,

   $\vec{L} = (0, 0, 2t^2)$,

   $\vec{a} = (-4t\cos(2t) - 4\sin(2t), -4t\sin(2t) + 4\cos(2t), 0)$,

   $\vec{\tau} = (0, 0, 4t)$;

   Vectors $\vec{r}$ and $\vec{v}$ are perpendicular at time $t = 0$.

2. Let trajectory be given by vector function $\vec{r}(t) = (\cos(2t), \sin(2t), t)$. Find velocity vector $\vec{v}(t) = \vec{r}'$, angular momentum vector $\vec{L}(t) = \vec{r} \times \vec{v}$, acceleration vector $\vec{a}(t) = \vec{r}''$ and torque vector $\vec{\tau}(t) = \vec{r} \times \vec{a}$.

   Answers $\vec{v} = (-2\sin(2t), 2\cos(2t), 1)$,

   $\vec{L} = (\sin(2t) - 2t\cos(2t), -\cos(2t) - 2t\sin(2t), 2)$,

   $\vec{a} = (-4\cos(2t), -4\sin(2t), 0)$,

   $\vec{\tau} = (4t\sin(2t), -4t\cos(2t), 0)$;

3. (a) Is the curve $\vec{r}(t) = (t^3, t^2, t)$ smooth?

   Answer Vector $\vec{v} = (3t^2, 2t, 1) \neq (0, 0, 0)$ for all t, and each its component is a continuous function of t. Thus the curve is smooth.

   (b) Is it true of false that the curve given by $\vec{r}(t) = (f(t), g(t), t)$ is smooth regardless of what the functions $f(t)$ and $g(t)$ are, as long as they are differentiable?

   Answer

   Continuity of f' and g' is required for the smoothness of the curve.

4. Use definition of the cross product and the product rule for differentiation to show that

$$(\vec{x} \times \vec{y})' = \vec{x}\,' \times \vec{y} + \vec{x} \times \vec{y}\,'.$$

   Solution Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$. Work out the formula for each component: take the derivative of the vector product $\vec{x} \times \vec{y}$ and rearrange into the sum of two vector products. For the first component we have

$$(x_2y_3 - x_3y_2)' = x_2'y_3 + x_2y_3' - (x_3'y_2 + x_3y_2') = (x_2'y_3 - x_3'y_2) + (x_2y_3' - x_3y_2').$$

   Similarly, for the second and the third.

$$(x_3y_1 - x_1y_3)' = x_3'y_1 + x_3y_1' - (x_1'y_3 + x_1y_3') = (x_3'y_1 - x_1'y_3) + (x_3y_1' - x_1y_3').$$

$$(x_1y_2 - x_2y_1)' = x_1'y_2 + x_1y_2' - (x_2'y_1 + x_2y_1') = (x_1'y_2 - x_2'y_1) + (x_1y_2' - x_2y_1').$$

5. Find the length of the curve

   (a) $\vec{r} = (3t^2, 12t, 8t^{3/2})$ where $0 \leq t \leq 2$.

   Answer The length is 36.

(b) $\vec{r} = (3t, 4\sin t, 4\cos t)$ where $-1 \le t \le 10$
Answer The length is 55.

6. Find the curvature and normal and binormal vectors at given point

(a) $\vec{r}(t) = (t, t^3, 0)$ at $(1, 1, 0)$
Answer First, point $(1, 1, 0)$ corresponds to $t = 1$. Curvature at $t = 1$ is $6/10^{3/2}$.
Unit tangent vector at $t = 1$ is $\vec{T} = 10^{-1/2}(1, 3, 0)$.
Normal vector at $t = 1$ is $\vec{N} = 10^{-1/2}(-3, 1, 0)$.
Binormal vector at $t = 1$ is $\vec{B} = (0, 0, 1)$.

(b) $\vec{r}(t) = (e^t, e^t \sin t, e^t \cos t)$ at $(1, 0, 1)$
Answer First, point $(1, 0, 1)$ corresponds to $t = 0$. Curvature at $t = 0$ is $\sqrt{2}/3$.
Unit tangent vector at $t = 0$ is $\vec{T} = 3^{-1/2}(1, 1, 1)$.
Normal vector at $t = 0$ is $\vec{N} = 2^{-1/2}(0, 1, -1)$.
Binormal vector at $t = 0$ is $\vec{B} = (-2, 1, 1)$.

7. Find and graph the osculating circle of the hyperbola $y = 2/x$ at point $(1, 2)$.
Answer The radius of the osculating circle is $R = 5\sqrt{5}/4$, and its center is at point $(7/2, 13/4)$.