Math 3202 **Solutions** **Assignment #5**

1. Evaluate the line integral of the vector field

   (a) $\int_C xy\,dx + (x-y)\,dy$, where C consists of line segments from (0,0) to (2,0) and from (2,0) to (3,2);

   Solution. First find line integral I_1 along segments from (0,0) to (2,0). Its parametric equation is $x = 2t$, $y = 0$, $0 \le t \le 1$. Thus $dx = 2dt, dy = 0$. We have $I_1 = 0$.

   Second find line integral I_2 along segments from (2,0) to (3,2). Its parametric equation is $x = t + 2$, $y = 2t$, $0 \le t \le 1$. Thus $dx = dt, dy = 2dt$. We have $I_2 = 17/3$. Total line integral is $I_1 + I_2 = 17/3$.

   (b) $\int_C \sin x\,dx + \cos y\,dy$, where C consists of the top half of the circle $x^2 + y^2 = 1$ from (1,0) to (-1,0) and line segments from (-1,0) to (-2,3).

   Answer. First line integral $I_1 = -\cos(-1) + \cos(1) = 0$.

   Second line integral $I_2 = \cos(1) - \cos(2) + \cos(3)$.

   Total integral is $I_1 + I_2 = \cos(1) - \cos(2) + \cos(3) \approx 1.1$.

2. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

   (a) $\vec{F} = (\,x^2y^3;\ -y\sqrt{x}\,)$, and $\vec{r} = (t^3; -t^3)$, $0 \le t \le 1$;

   Solution. $\vec{F}(\vec{r}(t)) = (-t^{15}; t^{9/2})$.

   $d\vec{r}(t) = \vec{v}(t)dt$, $\vec{v}(t) = (3t^2; -3t^2)$.

   $\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)dt = \int_0^1 -3t^{17} - 3t^{13/2}dt = -17/30$.

   (b) $\vec{F} = (\,x(x^2+y^2)^{-1/2};\ y(x^2+y^2)^{-1/2}\,)$, and $\vec{r} = (t, 1+t^2)$, $-1 \le t \le 1$.

   Answer. $\int_C \vec{F} \cdot d\vec{r} = 0$

3. Determine whether or not $\vec{F}$ is concervative vector field, and if yes, find the potential.

   (a) $\vec{F} = (2x\cos y - y\cos x; -x^2\sin y - \sin x)$;

   Answer. It is conservative. Potential $f(x,y) = x^2\cos y - y\sin x + C$.

   (b) $\vec{F} = (1 + 2xy + \ln x; x^2)$.

   Answer. It is conservative on its domain $x > 0$. Potential $f(x,y) = x^2y + x\ln x + C$.

4. Show that the line integral is path-independent and evaluate the integral along any path from (1,0) to $(2, \pi/4)$

   (a) $\int e^y\,dx + xe^y\,dy$;

   Solution. Show that the field is conservative and thus the line integral is path independent. For that show that $Q_y = P_x$ on the entire (xy)-plane. Then find potential $f(x,y) = xe^y + K$, and use it to evaluate integral

   $\int e^y\,dx + xe^y\,dy = f(2,\pi/4) - f(1,0) = 2e^{\pi/4} - 1$;

   (b) $\int \tan y\,dx + x\sec^2 y dy$.

   Solution. Show that the field is conservative on the open simply-connected region $-\pi/2 < y < \pi/2$, and thus the line integral in this region is path independent. For that show that $Q_y = P_x$ on the domain of $\vec{F}$. Then find potential $f(x,y) = x\tan y + K$, and use it to evaluate integral

   $\int \tan y\,dx + x\sec^2 y dy = f(2,\pi/4) - f(1,0) = 2.$;

5. Find the work done by force field $\vec{F} = (\,(y/x)^2;\ -2(y/x)\,)$ moving an object from (1,1) to (4,-2).

   Solution. Corresponding to the force $\vec{F}$ potential $f(x,y) = -y^2/x + C$. The force field is concervative on an open region $x > 0$. Thus work (=line integral) is path independent and is equal to $f(4,-2) - f(1,1) = 0$.

6. Consider vector field $\vec{F} = (\,-y(x^2+y^2)^{-1};\ x(x^2+y^2)^{-1}\,)$.

   a) Is $\vec{F}$ conservative?

   Solution: We check condition $Q_y = P_x = (y^2 - x^2)(x^2+y^2)^{-2}$.

   Nevertheless, the field is not concervative since its domain is not simply-connected region (has singularity at the origin).

   The field is concervative on any simply-connected region, which does not contain the origin.

   b) Calculate line integrals $\int_C \vec{F} \cdot d\vec{r}$ along lower and upper halfs of the circle $x^2 + y^2 = 1$ from (1,0) to (-1,0). Is the line integral path-independent?

   Solution:

   The upper half of the circle from (1,0) to (-1,0) is parametrized by $x = \cos t, y = \sin t, 0 \le t \le \pi$. The line integral is equal π.

   The lower part of the circle from (1,0) to (-1,0) is parametrized by $x = \cos t, y = -\sin t, 0 \le t \le \pi$. The line integral is equal $-\pi$.

   Thus the integral is path dependent.

   c) Is there a contradiction? Explain.

   Answer. No contradiction, because the field is not concervative in the entire plane, so path-independence was not expected.

   Notice, that the two paths taken in (b) surround the point of singularity. If the two paths were on the same side of the origin, the line integral would be path-independent.

7. Determine whether or not the given set is open, connected, simply-connected?

   a) $\{(x,y) |\ x < 0, y > 0\}$; Answer: open, connected, simply-connected.

   b) $\{(x,y) |\ |x| > 0\}$; Answer: open, not connected, not simply-connected.

   c) $\{(x,y) |\ x^2 + y^2 \le 5\}$; Answer: not open, connected, simply-connected.

   d) $\{(x,y) |\ x^2 + y^2 > 5\}$. Answer: open, connected, not simply-connected.