Math 3202 **Solutions** **Assignment #4, total=27 pt**

1. Find the mass and the center of mass of lamina that occupies region D and has density $\rho(x,y)$.

   Solution we use the formulas

$$m = \int\int_D \rho(x,y)\,dA, \quad \bar{x} = \frac{1}{m}\int\int_D x\rho(x,y)\,dA, \quad \bar{y} = \frac{1}{m}\int\int_D y\rho(x,y)\,dA,$$

   (a) $D = \{(x,y)|0 \le x \le a, 0 \le y \le b\}$, $\rho(x,y) = cxy$. Here a, b, c are positive constants.
   Answer
   $m = \int_0^a \int_0^b cxydydx = a^2b^2c/4$;
   $\bar{x} = \frac{1}{m}\int_0^a \int_0^b cx^2ydydx = 2a/3$;
   $\bar{y} = \frac{1}{m}\int_0^a \int_0^b cxy^2dydx = 2b/3$;

   (b) **[6pt]** D is the triangular region with vertices $(0,0)$, $(1,1)$, $(4,0)$; $\rho(x,y) = x$.
   Answer
   triangular region: $0 \le y \le 1$, $y \le x \le 4-3y$;
   $m = \int_0^1 \int_y^{4-3y} xdxdy = 10/3$;
   $\bar{x} = 0.3\int_0^1 \int_y^{4-3y} x^2dxdy = 2.1$;
   $\bar{y} = 0.3\int_0^1 \int_y^{4-3y} xydxdy = 0.3$;

   (c) **[6pt]** D is bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$; $\rho(x,y) = x$.
   Answer
   region: $0 \le y \le 1$, $y^2 \le x \le 1$;
   $m = \int_0^1 \int_{y^2}^1 x^2dxdy = 2/5$
   $\bar{x} = 5/2\ \int_0^1 \int_{y^2}^1 x^2dxdy = 5/7$
   $\bar{x} = 5/2\ \int_0^1 \int_{y^2}^1 xydxdy = 5/12$

   (d) D is the part of the disk $x^2+y^2 \le 1$ in the first quadrant. Density at point (x,y) is proportional to the square of the distance to the origine, $\rho(x,y) = c(x^2+y^2)$, where c is a positive constant.
   Answer
   change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.
   region in polar coordinates: $0 \le \theta \le \pi/2$, $0 \le r \le 1$;
   $m = \int_0^{\pi/2}\int_0^1 cr^2\,rdrd\theta = c\pi/8$;
   $\bar{x} = \int_0^{\pi/2}\int_0^1 cr^2\,rcos\theta\,rdrd\theta = 8/(5\pi)$;
   $\bar{y} = \int_0^{\pi/2}\int_0^1 cr^2\,rsin\theta\,rdrd\theta = 8/(5\pi)$;

2. Find the area of the surface:

   Solution: Let $z = f(x,y)$, $(x,y) \in D$.

$$A = \int\int_D \sqrt{1+f_x^2+f_y^2}\,dA.$$

   (a) the part of the plane $2x+5y+z = 10$, that lies inside the cylinder $x^2+y^2 = 9$.
   Answer: Here D is a circle of radius 3 with center at the origin; $z = 10-2x-5y$. Thus $A = \int_0^{2\pi}\int_0^3 \sqrt{1+4+25}rdrd\theta = 9\sqrt{30}\pi$.

(b) **[6pt]** the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.

Answer: xy-plane has equation $z = 0$. Thus here D is a circle of radius 2 with center at the origin. Consequently, $A = \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2} r dr d\theta = (17^{3/2} - 1)\pi/6$.

(c) the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Answer: Here D lies between circles of radii 1 and 2 with centers at the origin: $0 \le \theta \le 2\pi$, $1 \le r \le 2$;
$A = \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2} r dr d\theta = (17^{3/2} - 5^{3/2})\pi/6$.

(d) **[6pt]** the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

Answer: Rewrite equation of the sphere: $x^2 + y^2 + (z-2)^2 = 4$. The intersection curve is a circle: $z = 3$; $x^2 + y^2 = 3$. Thus domain D is $0 \le \theta \le 2\pi$, $0 \le r \le 3$;
and the surface of sphere inside the paraboloid has equation

$$z = 2 + \sqrt{4 - x^2 - y^2}.$$

Thus

$$A = \int_0^{2\pi} \int_0^2 \sqrt{1 + r^2/(4 - r^2)} r dr d\theta = 4\pi.$$

3. Evaluate

a) $\int_0^1 \int_x^{2x} \int_0^y 2xyz \, dz \, dy \, dx = 5/8$.

b) **[3pt]** $\int_0^1 \int_0^z \int_0^y ze^{-y^2} \, dx \, dy \, dz = (4e)^{-1}$.