Math 3202 **Answers** **Assignment #2**

1. **[3pt]** Show that if a particle moves with a constant speed, then the velocity and acceleration vectors are orthogonal.

   Solution. $v^2 = \vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2$

   Given $v_1^2 + v_2^2 + v_3^2$=const, differentiate both sides to get

$$v_1 \frac{dv_1}{dt} + v_2 \frac{dv_2}{dt} + v_3 \frac{dv_3}{dt} = 0.$$

   Recall that $\vec{a} = \dfrac{d\vec{v}}{dt}$. Thus, $\vec{v} \cdot \vec{a} = 0$, and they are orthogonal.

2. Use definition of the cross product and the product rule for differentiation to show that

$$(\vec{a} \times \vec{b})' = \vec{a}\,' \times \vec{b} + \vec{a} \times \vec{b}\,'.$$

   Solution.

   Let $\vec{a} = (x_1, x_2, x_3)$ and $\vec{b} = (y_1, y_2, y_3)$. Work out the formula for each component: take the derivative of the vector product $\vec{a} \times \vec{b}$ and rearrange into the sum of two vector products. For the first component we have

$$(x_2y_3 - x_3y_2)' = x_2'y_3 + x_2y_3' - (x_3'y_2 + x_3y_2') = (x_2'y_3 - x_3'y_2) + (x_2y_3' - x_3y_2').$$

   Similarly, for the second and the third.

$$(x_3y_1 - x_1y_3)' = x_3'y_1 + x_3y_1' - (x_1'y_3 + x_1y_3') = (x_3'y_1 - x_1'y_3) + (x_3y_1' - x_1y_3').$$

$$(x_1y_2 - x_2y_1)' = x_1'y_2 + x_1y_2' - (x_2'y_1 + x_2y_1') = (x_1'y_2 - x_2'y_1) + (x_1y_2' - x_2y_1').$$

3. Find the tangential and normal components of the acceleration vector if the position is given by

   a)**[3pt]** $\vec{r} = (3t - t^3;\ 3t^2; 0)$

   Solution.

   Velocity $\vec{v} = (3 - 3t^2;\ 6t; 0)$;

   Acceleration $\vec{a} = (-6t;\ 6; 0)$;

   Speed $v(t) = 3t^2 + 3$, thus $a_T = dv/dt = 6t$.

$$a_N = \frac{|a_1v_2 - a_2v_1|}{v} = \frac{18t^2 + 18}{3t^2 + 3} = 6.$$

   b) $\vec{r} = (e^t;\ \sqrt{2}t;\ e^{-t})$

   Answer. $a_T = e^t - e^{-t}$; $a_N = \sqrt{2}$.

4. **[6pt]** Find the tangential, normal, and binormal vectors for curve $\vec{r}(t) = (e^t, e^t \sin t, e^t \cos t)$ at point $(1, 0, 1)$.

   Answer. Note, the point corresponds $t = 0$.

$$\vec{T}(t) = \frac{1}{\sqrt{3}}(1, \sin t + \cos t, \cos t - \sin t)$$

$$\vec{T}(0) = \frac{1}{\sqrt{3}}(1, 1, 1)$$

$$\vec{N}(t) = \frac{1}{\sqrt{2}}(0, -\sin t + \cos t, -\cos t - \sin t)$$

$$\vec{N}(0) = \frac{1}{\sqrt{2}}(0, 1, -1)$$

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \frac{1}{\sqrt{6}}(-2, 1, 1).$$

5. **[3pt]** Find equations of **normal plane** and osculating plane for the curve $\vec{r} = (t;\, t^2;\, t^3)$ at point (1,1,1).

   Answer.

   $t = 1$. $v(1) = (1, 2, 3)$. Normal plane $x + 2y + 3z = 6$.

   Osculating plane $3x - 3y + z = 1$.

6. **[3pt]** Find osculating circle of the ellipse $9x^2 + 4y^2 = 36$ at points $(2, 0)$ and $(0, 3)$.

   Answer. The parametric equation of the ellipse can be writted as $x = 2\cos t$, $y = 3\sin t$, $t \in [0, 2\pi]$. Then curvature $\kappa = 6(4 + 5\cos^2 t)^{-3/2}$.

   Thus at point $(2, 0)$ $(t = 0)$ $\kappa = 2/9$. Oscullating circle has radius $R = 9/2$ and center at $(-2.5,\ 0)$.

   At point $(0, 3)$ $(t = \pi/2)$ $\kappa = 3/4$. Oscullating circle has radius $R = 4/3$ and center at $(0,\ 5/3)$.

7. Derive the formula for the arc length

$$l = \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

   for an arc given by $\vec{r}(t) = (x(t); y(t); z(t))$, $t \in [0, T]$.

   Answer. $l = \int_0^T v(t)dt$, and $v(t) = \sqrt{\vec{v}(t) \cdot \vec{v}(t)}$, and $\vec{v}(t) = d\vec{r}(t)/dt$.