Math 3001 **Due Fri Nov 25** **Assignment #9**

1. Prove Abel's theorem:

   Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R = 1$, and let $\sum_{n=0}^{\infty} a_n$ be convergent. Then

$$\lim_{x\to 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

2. Use the statement obtained in Problem 1 and Cauchy's theorem about multiplication of two absolutely convergent series to show that

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \left(\sum_{n=0}^{\infty} c_n\right), \quad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

   if all there series converge (not necessarily absolutely).

   P.S. This statement was published by Abel in 1826.

3. A) For which values of $x \in \mathbf{R}$ the sequence

$$S_n = |\sum_{k=1}^{n} \cos(kx)|$$

   is bounded?

   B) Let sequence $\{f_n\}_{n=1}^{\infty}$ be monotone and $\lim_{n\to\infty} f_n = 0$. For which values of $x \in \mathbf{R}$ the trigonometric series $\sum_{n=1}^{\infty} f_n \cos(nx)$ converges?

4. A) Explain why series

$$\sum_{n=1}^{\infty} \frac{-(-1)^n \sin(nx)}{n}$$

   converges uniformly on $(-\pi + \delta, \pi - \delta)$ for any $0 < \delta < \pi$.

   B) Show that the series in (A) is the Fourier series for function $F(x) = x/2$ on $(-\pi, \pi)$.

   C) Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin(5\pi n/4)}{n}$$

5. Give an example of a function for which corresponded to it Fourier series has only finite number of terms.

6. EXTRA POINTS We have seen that the function defined as $F(x) = e^{-x^{-2}}$ for $x \neq 0$ and $F(x) = 0$ for $x = 0$ is not equal to its Taylor series centered at zero for all $x \neq 0$.

   Consider now Taylor series centered at $a \neq 0$ for this function. Can you use it to evaluate $F(0)$?