Math 3001 **Due Wed Nov 9**

Assignment #7

1. Find a function to which the series converges pointwise. Does it converge uniformly on the whole domain?

A)

$$\frac{x^2}{x^2+1}+\frac{x^2}{(x^2+1)(2x^2+1)}+\frac{x^2}{(2x^2+1)(3x^2+1)}+\cdots+\frac{x^2}{(nx^2+1)((n+1)x^2+1)}+\cdots.$$

Solution:
The above series can be rewritten as

$$\sum_{n=0}^{\infty}\frac{x^2}{(nx^2+1)((n+1)x^2+1)}=\sum_{n=0}^{\infty}\left[\frac{1}{nx^2+1}-\frac{1}{(n+1)x^2+1}\right].$$

Notice that this is a telescoping series. Let S_n be the n^{th} partial sum of this series. Then

$$S_n=1-\frac{1}{(n+1)x^2+1}.$$

If $x=0$, we have $S_n=0$ for all n, thus $S(x)=0$. If $x\neq 0$, then $\lim\limits_{n\to\infty} S_n=1$. Therefore, on $D=\mathbb{R}$,

$$\sum_{n=0}^{\infty}\frac{x^2}{(nx^2+1)((n+1)x^2+1)}\longrightarrow S(x)=\begin{cases}0 & \text{if } x=0\\ 1 & \text{otherwise}\end{cases}.$$

This convergence is not uniform on $\mathbb{R}$ since $f_n(x)$ is continuous on $\mathbb{R}$ for all n and x, but $f(x)$ is discontinuous at $x=0$.

B)

$$\sum_{n=0}^{\infty}\frac{x^5}{(x^5+1)^n}.$$

Solution:
Now this series can be rewritten as $x^5\sum\limits_{n=0}^{\infty}\left(\frac{1}{x^5+1}\right)^n$. Since $\frac{1}{x^5+1}<1$

for all $x \in [0, \infty)$, we have a convergent geometric series with $r = \dfrac{1}{x^5+1}$. Therefore, for $D = (0, \infty)$, we have

$$x^5 \sum_{n=0}^{\infty} \left(\frac{1}{x^5+1} \right)^n = x^5 \left(\frac{1}{1 - \frac{1}{x^5+1}} \right) = x^5 + 1 = S(x).$$

If $x = 0$, then $S(x) = 0$. Thus, on $D = [0, \infty)$,

$$x^5 \sum_{n=0}^{\infty} \left(\frac{1}{x^5+1} \right)^n \longrightarrow f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^5 + 1 & \text{otherwise} \end{cases}.$$

This convergence is not uniform on $D = [0, \infty)$ since $f_n(x)$ is continuous on D for all n and x, but $f(x)$ is discontinuous at $x = 0$.

2. Prove a Theorem that if a sequence of continuous functions converges uniformly on a set S to a function f, then f is continuous on S.

   Proof. Let $\{ f_n(x) \}_1^\infty$ denote a sequence of continuous functions on a set S. So for all $\varepsilon > 0$ and for all $c \in S$, there exists $\delta_1 > 0$ such that for all $x \in S$,

   $$0 < |x - c| < \delta_1 \text{ implies } |f_n(x) - f_n(c)| < \frac{\varepsilon}{3}.$$

   Assume $f_n(x)$ converges uniformly on S to $f(x)$. Then for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$ and for all $x \in S$ we have $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$.

   Now fix $\varepsilon > 0$ and $c \in S$. Let $\delta = \delta_1$ and $N = N_1$. Then for all $n > N$ and for all $x \in S$, when $|x - c| < \delta$ we have

   $$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon \end{aligned}$$

   Therefore, $f(x)$ is continuous on S. $\square$

3. Using the Theorem about uniform convergence and continuity, determine whether or not the following sequence of functions converges uniformly on $x \in [0, 1]$

   A)

   $$f_n(x) = \frac{5}{3x^n + 4}$$

   Solution:
   If $x = 1$, we have $f_n(x) = \frac{5}{7} = f(x)$. If $x \neq 1$ we have $\lim_{n\to\infty} f_n(x) = \frac{5}{4}$. Thus, on $D = [0, 1]$, we have

   $$f_n(x) \longrightarrow f(x) = \begin{cases} \frac{5}{7} & \text{if } x = 1 \\ \frac{5}{4} & \text{otherwise} \end{cases}$$

   This convergence is not uniform on $D = [0, 1]$ since $f_n(x)$ is continuous on D for all n and x, but $f(x)$ is discontinuous at $x = 1$.

   B)

   $$f_n(x) = \frac{nx}{nx + 4}$$

   Solution:
   If $x = 0$, we have $f_n(x) = 0 = f(x)$. If $x \neq 0$ we have $\lim_{n\to\infty} f_n(x) = 1$. Thus, on $D = [0, 1]$, we have

   $$f_n(x) \longrightarrow f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

   This convergence is not uniform on $D = [0, 1]$ since $f_n(x)$ is continuous on D for all n and x, but $f(x)$ is discontinuous at $x = 0$.

4. Find a sequence of functions (f_n) defined on $[0, 1]$ such that each f_n is discontinuous at each point of $[0, 1]$ and such that the sequence converges uniformly to a function f that is continuous on $[0, 1]$.

   Solution:

   $$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

   This function is discontinuous everywhere and converges uniformly on $[0, 1]$ to 0.

5. Evaluate (with explainations)

$$\lim_{n\to\infty}\int_0^{\pi}\frac{n+\sin(nx)}{3n+\sin^2(nx)}\,dx.$$

Solution:

Since $\lim_{n\to\infty}\dfrac{n+\sin(nx)}{3n+\sin^2(nx)}=\dfrac{1}{3}$ on $D=[0,\pi]$, we check whether or not

$\dfrac{n+\sin(nx)}{3n+\sin^2(nx)} \longrightarrow \dfrac{1}{3}$ uniformly on $D=[0,\pi]$. So fix $\varepsilon>0$. Choose N such that $N>\dfrac{1}{3\varepsilon}$. Then for all $n>N$ and for all $x\in[0,\pi]$, we have

$$\left|\frac{n+\sin(nx)}{3n+\sin^2(nx)}-\frac{1}{3}\right| \le \left|\frac{n+1}{3n}-\frac{1}{3}\right| = \frac{1}{3n} < \frac{1}{3N} = \varepsilon$$

So we now have

$$\lim_{n\to\infty}\int_0^{\pi}\frac{n+\sin(nx)}{3n+\sin^2(nx)}\,dx = \int_0^{\pi}\frac{1}{3}\,dx = \frac{\pi}{3}$$

6. Integrate the geometric series

$$\frac{1}{1-r}=1+r+r^2+r^3+\cdots$$

term by term from $-x$ to x, where $0<x<1$ to find the series for $\ln\frac{1+x}{1-x}$. Explain the reason why you can do this.

Solution:

Let $0<x<1$. Then we know that the series $\displaystyle\sum_{n=0}^{\infty} r^n$ converges uniformly on $D=[-x,x]$ to $\dfrac{1}{1-r}$. Therefore, we have

$$\int_{-x}^{x}\frac{1}{1-r} = \int_{-x}^{x}\sum_{n=0}^{\infty} r^n\,dr$$

Integrating both sides gives us

$$\frac{\ln(1+x)}{\ln(1-x)} = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

7. EXTRA POINTS

   If possible, find a sequence of functions (f_n) defined on $[0,1]$ such that each f_n is continuous at each point of $[0,1]$ and such that the sequence converges uniformly to a function f that is discontinuous at each point of on $[0,1]$.

   Solution:
   This is not possible by question # 2.