Math 3001 **Due Wed Oct 26** **Assignment #6**

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. Prove that

   A)

$$\text{liminf}\frac{a_{n+1}}{a_n} \leq \text{liminf}\, a_n^{1/n} \leq \text{limsup}\, a_n^{1/n} \leq \text{limsup}\frac{a_{n+1}}{a_n}$$

   Solution. To prove the inequality on the right, let $\alpha = \text{limsup}\frac{a_{n+1}}{a_n}$. If $\alpha = \infty$ the result is obvious. If α is finite, choose $\beta > \alpha$. Then there exists N such that $\frac{a_{n+1}}{a_n} < \beta$ for all $n > N$. That is for $n > N$ we have

$$a_n < \beta a_{n-1}, \quad a_{n-1} < \beta a_{n-2}, \qquad a_{N+1} < \beta a_N.$$

   Combine these inequalities to obtain $a_{N+k} < \beta^k a_N$, (for any $k \geq 1$) or equivaletly, that $a_n < c\beta^n$, where $c > 0$ is a constant. Thus $\text{limsup}(a_n)^{1/n} \leq \beta$. Since this holds for every $\beta > \alpha$, the desired inequality follows.

   B)

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

   provided they both exist;

   Solution. Recall, that if limit exists them limsup=liminf. Thus B) follows from A) in this case.

2. Give (if possible) an example of a power series which has the following interval of convergence

   a) (-1,1]

   Answer $\sum \frac{(-1)^n}{n} x^n$

   b) [-1, 1)

   Answer $\sum \frac{1}{n} x^n$

   c) $(-1/2, 0) \cup (0, 1/2)$

   Answer Not possible.

   d) [2,4]

   Answer $\sum \frac{1}{n^2}(x-3)^n$

3. Let R be the radius of convergence for the power series $\sum a_n x^n$. If infinitely many of the coefficients a_n are nonzero integers, prove that $R \leq 1$.

   Solution. If there are infinitly many coefficients such that $|a_n| \geq 1$ then α =limsup$|a_n|^{1/n} \geq 1$. Thus $R = 1/\alpha \leq 1$.

4. Suppose that the series $\sum a_n$ diverges, but the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded. What can you say about the radius of convergence of the power series $\sum a_n x^n$?

   Solution. The radius is equal to 1. The proof goes as follows:

   1. Since the series $\sum a_n$ diverges the radius can't be greater then 1.

   2. Assume the radius is less then 1.

   Then α =limsup$\frac{a_n}{a_{n+1}} < 1$. Thus for any β such that $\alpha < \beta < 1$ there is N so that $\frac{a_n}{a_{n+1}} < \beta$ for all $n \geq N$.

But then $a_n < \beta a_{n+1}$ for all $n \geq N$. Thus

$a_N < \beta^k a_{N+k}$ for all $k \geq 1$. Here a_N is a constant.

So we get const$\beta^{-k} < a_{N+k}$. Since $\beta < 1$, the sequence $\{a_n\}$ is shown to be unbounded, which is a contradiction. Thus the radius is not less then 1, but is equal to 1.

5. Prove that the series $\sum a_n x^n$ and $\sum n a_n x^n$ have the same radius of convergence.

   Solution. $R = \text{limsup}\frac{a_n}{a_{n+1}} = \text{limsup}\frac{n a_n}{(n+1)a_{n+1}}$.

   Here we use the following Theorem:

   If sequence $\lim_{n\to\infty} r_n = r$ and s_n is a bounded sequense then

   $$\text{limsup} s_n r_n = r \text{limsup} s_n.$$

   In our case $r_n = \frac{n}{n+1}$.

6. Let $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for $x \in R$.

   a) Show that (f_n) converges uniformly to f.

   Solution. For every x, $\lim_{n\to\infty} x + \frac{1}{n} = x$. Thus (f_n) converges to f pointwise.

   Now, $|f_n - f| = \frac{1}{n}$. This difference is independent from x and thus for every ϵ there is N such that $|f_n - f| < \epsilon$ for $n > N$ and for all x at the same time. Thus the convergence is uniform.

   b) Show that $(f_n)^2$ converges pointwise to f^2, but not uniformly.

   Solution. For every x, $\lim_{n\to\infty}(x + \frac{1}{n})^2 = x^2$. Thus (f_n^2) converges to f pointwise.

   Now, $|f_n^2 - f^2| = \frac{1}{n^2} + \frac{2x}{n}$.

   Since x arbitrarily large, there is to such a number N same for all x that $|f_n^2 - f^2|$ would be less then ϵ for $n > N$. This number N depends on x, and thus the convergence in not uniform.

   c) Is it true or false that $(f_n)^2$ converges uniformly to f^2 on any finite segment, i.e. $x \in [a, b]$.

   Answer Yes, it is true, considering previous discussionin b).

7. Give an example (if possible) of a sequence of functions $f_n(x)$ pointwise convergent to $f(x)$ on (a, b) such that

   a) all $f_n(x)$ are continuous, but $f(x)$ is not.

   Answer $f_n = x^n$, $0 \leq x \leq 1$.

   b) $\lim_{n\to\infty} f_n'(x) \neq f'(x)$.

   Answer $f_n = (\sin(nx))/\sqrt{n}$.

   c) $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b f(x)\,dx$.

   Answer $f_n = n^2 x$ for $0 \leq x \leq 1/n$, $f_n = 0$ for $1/n \leq x \leq 1$.

8. EXTRA POINTS

   Suppose (f_n) converges poinwise to f on a set S. Prove that (f_n) converges uniformly to f on every **finite** subset of S.

   Answer On a finite set it is always possible to find N independent of the members of the set. Just take the maximum of all individual $N(x)$ w.r.to x.