Math 3001 **Due Fri Oct 7** **Assignment #4**

1. Let f, g be integrable on $[a, b]$. Introduce notations

$$||f|| = (\int_a^b f^2(x)dx)^{1/2}, \quad (f, g) = \int_a^b f(x)g(x)dx$$

   a) Prove Cauchy-Swartz inequality $|(f, g)| \le ||f|| \cdot ||g||$.

   b) Show that Cauchy-Swartz inequality implies the *triangle inequality*

$$||f + g|| \le ||f|| + ||g||.$$

2. Find the second derivative $F''(x)$

   a) $F(x) = \int_0^{\sin x} \cos(t^2)dt$

   b) $F(x) = \int_{-x}^{x^2} \sqrt{1 + t^2}dt$

   c) $F(x) = \int_0^x xe^{t^2} dt$

3. Evaluate $\lim_{x\to 0}(x^{-1} \int_0^x \sqrt{9 + t^2}dt)$

4. Let f be continuous on $[a, b]$. Suppose $\int_a^x f(t)dt = \int_x^b f(t)dt$ for all $x \in [a, b]$. Find function f.

5. Let f be continuous on $[0, \infty)$. Let $f(x) \ne 0$ for $x > 0$ and $f^2(x) = 2\int_0^x f(t)dt$. Find function f.

6. Let $I_n = \int_0^\infty x^{-n}dx$. For which real values n the integral I_n is convergent?

   Hint: consider separately $\int_0^1 x^{-n}dx$ and $\int_1^\infty x^{-n}dx$

7. Is the following argument correct? Explain.

   $\int_{-L}^{L} \sin x dx = 0$ for any $L \ge 0$. Thus $\int_{-\infty}^{\infty} \sin x dx = 0$.

8. Prove the following statement:

   Let f be continuous on $[a, b]$ and g be continuous on $[c, d]$, where $f([a, b]) \subset [c, d]$. Then the composition $g \circ f$ is integrable on $[a, b]$.

9. **Extra Points Problem**

   Prove the following statement:

   Let f be integrable on $[a, b]$ and g be continuous on $[c, d]$, where $f([a, b]) \subset [c, d]$. Then the composition $g \circ f$ is integrable on $[a, b]$.