Math 3001 **Due Fri Sept 30**

Assignment #3

1. Consider piece-wise constant function on $[0,1]$ defined by formula

$$F(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, \quad n = 0, 1, 2, ... \quad F(0) = 0.$$

Explain why this function is integrable and find the integral $\int_0^1 F(x)dx$.

Solution

Clearly this function is monotonic, hence it is integrable on $[0,1]$. For $n = 0$, we have the area under the interval $[1/2, 1] = (1)(1/2)$. For $n = 1$, we have the area under the interval $[1/4, 1/2] = (1/2)(1/4)$. For $n = 2$, we have the area under the interval $[1/8, 1/4] = (1/4)(1/8)$. Continuing, we have

$$\int_0^1 F(x)\,dx = \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i = \frac{1}{2}\left(\frac{1}{1-1/4}\right) = \frac{2}{3}.$$

$\square$

2. Show that if $f(x) \leq g(x)$ for all $x \in [a,b]$, and are integrable functions, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Solution

Consider $h(x) = g(x) - f(x)$. Observe that $h(x) \geq 0$ and it is integrable for all $x \in [a,b]$. Then using the definition of the Riemann integral show that $\int_a^b h(x)\,dx$ can't be negative and thus $\int_a^b h(x)\,dx \geq 0$. It implies by linearity that $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

3. Let f be integrable on $[a,b]$ and suppose that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Show that $m(b-a) \leq \int_a^b f dx \leq M(b-a)$.

Solution

From question #2, we have that $\int_a^b m \leq \int_a^b f(x)\,dx \leq \int_a^b M$. Therefore

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

$\square$

4. Prove the mean value theorem for integrals: *It f is continuous on $[a, b]$ then there exists $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f dx$$

Solution
Since f is continuous on $[a, b]$, it is bounded on $[a, b]$. Let $M = \max\{f(x) \mid x \in [a, b]\}$ and $m = \min\{f(x) \mid x \in [a, b]\}$. Then we have $m \leq f(x) \leq M$. From question #3, we have $m \leq \frac{1}{b-a} \int_a^b f dx \leq M$. Therefore, by the **Intermediate Value Theorem**, there must exist $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f dx$. $\square$

5. Let f and g be continuous on $[a, b]$, and suppose that $\int_a^b f\, dx = \int_a^b g\, dx$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = g(c)$.

Solution
Let $h(x) = f(x) - g(x)$ for all $x \in [a, b]$. So we have $h(x) = 0$ for all $x \in [a, b]$. Thus, $\frac{1}{b-a} \int_a^b f dx = 0$. Therefore, from question #4, there exists $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f dx = 0$, i.e. there exists $c \in [a, b]$ such that $f(c) - g(c) = 0 \Rightarrow f(c) = g(c)$.$\square$

6. Let f be integrable function on $[a, b]$ and let k be any real number. Show that kf is also integrable and that $\int_a^b kf(x)\, dx = k \int_a^b f(x)\, dx$.

Hint: consider cases $k = 0$, $k > 0$, $k < 0$.

Solution
We consider two cases. Let P be a partition of $[a, b]$. For $k \geq 0$, we have

$$U(kf) = kU(f) = kL(f) = L(kf).$$

Therefore, kf is integrable. Also

$$\int_a^b kf(x)\, dx = U(kf) = kU(f) = k \int_a^b f(x)\, dx.$$

For $k < 0$, we have

$$U(kf) = kL(f) = kU(f) = L(kf).$$

Therefore, kf is integrable. Also,

$$\int_a^b kf(x)\,dx = U(kf) = kL(f) = k\int_a^b f(x)\,dx.$$

$\square$

7. Show if $a < c < b$ then for any integrable function f

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Solution
Since f is integrable on $[a,b]$, there exists partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$ for all $\epsilon > 0$. Let P_1 be points of P which lie to the left of, and include, c. Let P_2 be points of P which lie to the right of, and include, c. Then $P = P_1 \cup P_2$. Now, $U(f,P_1) - L(f,P_1) \leq U(f,P) - L(f,P) < \epsilon$. Hence, f is integrable on $[a,c]$. Similarly, f is integrable on $[c,b]$.

Since $U(f,P) - L(f,P) \;=\; U(f,P_1) + U(f,P_2) - L(f,P_1) - L(f,P_2) < \epsilon$

we have

$$\begin{aligned} \textstyle\int_a^b f \leq U(f,P) &= U(f,P_1) + U(f,P_2) \\ &< L(f,P_1) + L(f,P_2) + \epsilon \leq \textstyle\int_a^c f + \int_c^b f - \epsilon \end{aligned}$$

and also

$$\begin{aligned} \textstyle\int_a^b f \geq L(f,P) &= L(f,P_1) + L(f,P_2) \\ &> U(f,P_1) + U(f,P_2) - \epsilon \leq \textstyle\int_a^c f + \int_c^b f + \epsilon \end{aligned}$$

Since this is true for all ϵ, we have $\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

8. Is it true or false that *if* $|f|$ *is integrable on* $[a,b]$ *then* f *must be integrable on* $[a,b]$?

Solution
False. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

9. **Extra Points Problem**

a) Consider function $f(x) = \sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Is that function integrable on $[0, 1]$? Explain.

b) Let f be bounded on $[a, b]$ and integrable on $[c, b]$ for any $c \in (a, b)$. Is such a function integrable on $[a, b]$?

Solutions

(a) f is integrable on $[0, 1]$ since it is continuous everywhere except at point $x = 0$ and bounded on $[0, 1]$.

(b) Yes.