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A NOTE ON MATHEMATICAL THINKING AND SCHOOL TERMINOLOGY

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Logical thinking and reasoning are the processes emphasized by contemporary mathematical curricula in many countries. However, sometimes problem composers' fear that students are not ready to handle certain mathematical situations becomes an obstacle on the way of the students' natural progress. This is especially regretful when an artificially simplified construction prevents the learners from developing genuine mathematical concepts where it could be done with just a little more thoughtful approach to the matter. In this paper we discuss a pseudo-simplified way in which the notion of a 'region of uniform width surrounding a rectangle' is treated in the secondary school problems. We found evidence that secondary school students are able to handle the model consistent with the standard mathematical definition. Moreover, scaffolding of a learner towards such a model contributes positively to the development of their mathematical thinking. We suggest that in order to fulfil the agenda of teaching students to reason logically, more attention to seemingly little details in mathematical textbooks is needed.

Keywords: Mathematical reasoning, Region of uniform width, Formal definitions, Secondary school teaching, Uniform neighbourhood.

Learning Terminology and Formal Definitions

Formal definitions play a very important role in rigorous mathematical thinking. However, many educators would argue that in teaching grade school mathematics one cannot start with formal definitions (Tall et al, 2012). Instead, more intuitive approaches, experiences and explorations should precede and support formal considerations. Poincare (1996/1914) compares formal definition of a circle with the naïve statement that "a circle is a round". The naive definition is less precise from a formal mathematical point of view but it is more insightful and accessible from the learner's standpoint. By starting to work with naïve definitions, the students will gradually come to understanding of their limitations and thus to the necessity to describe the concepts more precisely and formally.

One of the most suitable moments for introducing new terminology is when the learner has noticed and singled out a certain phenomenon in the course of their experience and naturally calls for a name for it. This student-centered scenario often requires a differentiated instruction for its implementation (Small, 2009). Alternatively, new terminology is introduced by the teacher who gives some reasonably formal definitions accompanied by motivations, explanations, and demonstrations of examples and non-examples. Gradually the definition may develop into more formal and abstract. In the case of a circle, the students will eventually come to the notion of a circle as a 'set of all points in the plane equidistant from a given point' and ultimately express this in algebraic notation as $\{(x,y) \mid (x-x_0)^2+(y-y_0)^2=R^2\}$ for a circle with radius R and center (x_0, y_0).

This approach seems to be very practical but not without a pitfall. Specifically, when it reduces solely to simplified examples and explanations that may even be incorrect from a higher mathematics viewpoint. Such explanations in fact do a disservice because they prompt the students to form wrong notions and misconceptions at an early stage of their study. In this connection I believe that instead of imposing an oversimplified definition on the students they would better be given some freedom to negotiate various possibilities and decide what level of complexity they can handle. In the next section I give an example of such situation.

The Notion of 'Region of Uniform Width' and Its Treatment in Grade School

The adjective 'uniform' in everyday language means in various contexts either 'identical' or 'consistent', as from example to example, place to place, or moment to moment, or 'constant', or 'without variations'. For example, one can say "a ribbon has uniform width" meaning that the ribbon has the same width all along regardless where it is measured. The width of a stick would be regarded as `non-uniform' if for example it is thick at one end and tapers down to thin at the other. In mathematics, 'uniform with respect to the variable x' generally means 'independent of x', but details must be clarified in the context. In order to illustrate this idea, let us analyze the statement "rectangle ABCD with dimensions $W \times L$ has uniform width W with respect to $0 \le x \le L$." Consider a coordinate plane and place the rectangle so that its corners have the coordinates A(0,0), B(L,0), C(L,W), D(0,W). The width of the rectangle at horizontal position x is the distance between the points (x,0) and (x,W). This distance equals W regardless of the value of x between 0 and L. Therefore, we say that the rectangle ABCD has uniform width W. This seems to be an obvious idea until we start to consider more elaborate situations. For instance, how should we define the region of uniform width around a given object? In the sequel, we focus on the notion of region of uniform width around a given rectangle, which is commonly used in secondary school mathematical problems.

In an online mathematical forum (Mymathforum, 2011) the following problem entitled "Find the width of the uniform border" was posted on November 1, 2011 at 4:26 pm:

> *A mural is to be painted on a wall that is 15 units long and 12 units high. A border of uniform width is to surround the mural. If the mural is to cover 75% of the area of the wall, how wide must the border be?*

Here "uniform width" is used in the context of painting, which suggests a familiar image of a picture and its frame. Consequently, 'border of uniform width' is perceived as a region bounded by two nested rectangles (see Figure 1, left), the outer one representing the entire wall, and the solution follows. The response posted on the same website just six minutes later reflects this understanding of the region of uniform width around a rectangle:

> *The "75%..." part is the easiest to tackle. The wall has area 12 x 15 = 180 (square units), so we want our mural to have area 0 .75*180 = 135 (square units). Now we determine the dimensions of the mural. The requirement of a "uniform border" means that the mural will have dimensions (15 - 2x)*(12 - 2x), where x is the width of the border. Deducing this is often the hardest part of problems like this, so draw a picture and see why my claim is true. So (15 - 2x)*(12 - 2x) = 135 is the equation for the area of the mural. FOIL, get in standard form, then factor/quadratic formula.*

This quick response confirms that the problem is quite typical as well as the given solution. Note that drawing is not even discussed – it is assumed that the region is the one stated above. Surely, it is not very surprising in the problem about a picture frame and similar real-world objects that presuppose the same image of nested rectangles. Here is another problem and its solution (Answerbag, 2008) posted on January 27, 2008:

> Question: *A walkway of uniform width has area 72 m² and surrounds a swimming pool that is 8 m wide and 10 m long. Find the width of the walkway.*
>
> Answer: *Call the width W. The pool forms an inner 8x10 rectangle, surrounded by an outer rectangle formed by the outside of the walkway. The length of this outer rectangle is 10+2W and the width is 8+2W. The area of the walkway itself is the difference between the areas of the two rectangles (a diagram would help here). So the equation is:*
> *(2W+8)*(2W+10) - 8*10 = 72. Can you work out the rest? Good luck!*

Another website Kwiznet (2012) contains quizzes for grade school mathematics. Under the title "Area of Rectangular Path – III" we find the following example that is of interest (bold is original):

> Example: *A path of uniform width runs around and outside a square plot of side 20 metres. If the area of the path is 276 square metres find its width.*
>
> Solution: *Let the width of the path be x metres. Now each side of the given square plot = 20m*
> *Therefore, each side of the outer square = (20 + 2x) metres, because the path runs outside.*
> ***The area of the path = (The area of outer square) - (The area of the inner square)***
> *= $(20 + 2x)^2 - 20^2$*
> *But the area of the path is given to be 276 square metres.*
> *Therefore, $(20 + 2x)^2 - 20^2 = 276$.*

After detailed calculations the answer is found to be '3 metres'. This example is accompanied by Directions that start as follows:

> *Read the above example and answer the question: Draw a square plot of side 12 cm and label its sides. Draw a path of uniform width around and outside the square plot …*

At this point it is assumed that the solver (in order to conform to the given solution!) will draw two nested squares. This last example differs from two previous examples by the fact that it refers to a plot, not a swimming pool or a picture frame. We do not have to deal with a rigid image of two nested rectangles automatically. Yet the solution introduces two nested squares, again without any discussion about a possible alternative. Thus we see that the model of two nested rectangles is constantly used in various situations to represent a 'region of uniform width around a rectangle'. While being appropriate for many explicit contexts such model disagrees with mathematical notion of region of uniform width. The next section introduces an alternative model as well as a problem from which it naturally arises.

A Logical Development of the Notion of 'Region of Uniform Width'

During the Fall of 2011 the author of this paper was meeting informally with several students interested in mathematics and problem solving. At one of those meetings, the subject of the region of uniform width was brought to attention.

Two students, call them Ron and Jon, were independently asked to draw a region of uniform width around a rectangle. Ron took a ruler and immediately started to draw two nested rectangles. He was a grade 11 student undoubtedly familiar with the kind of problems discussed in the previous section. Even though the context was not specified, Ron knew the 'standard' answer to the question. He did not hesitate for a moment and put all his effort to produce a neat figure.

Jon being a much younger student just entering junior high level was not yet spoiled by the stereotype. He started to draw the region and at first produced two nested rectangles by drawing segments parallel to each side of the inner rectangle and extending them until the intersection. Then he looked at his figure and said that he was not sure about the corners. After a minute he cut the corners of the exterior rectangle transforming it into an octagon (see Figure 1, middle). When asked why he decided to cut the corners, Jon replied, “in case of nested rectangles the distance between the corner points of the inner and exterior rectangles is bigger than the distance between the sides and thus the width of the region is not uniform”. It is evident that Ron was using a cognitive schema developed from seeing a number of examples that in his mind stick with the phrase “region of uniform width around a rectangle”. On the other hand, Jon tried to apply the meaning of the word ‘uniform’ and an unbiased logical thinking in order to build the notion of the ‘region of uniform width around a rectangle’. While he was not successful at once, he nevertheless was able to see a problem with the nested rectangles model.

Notably, when Ron was asked to rethink his construction and to check if his region indeed had the uniform width everywhere, corners including, he suddenly got perplexed. He then claimed that it is impossible to build such a region around a rectangle. He explained that a model of a region of uniform width that he now had in mind, was “the trace of two parallel wheels of a car. But if the car turns around then the traces are either two nested circles or ovals, in any case it is a curve without sharp corners.” Thus, he said, “to have a band of uniform width around it, the inner region can’t be a rectangle.”

Now that both students got stuck with their sketches they were given the following problem:

> *Ants are confined under a rectangular lid ABCD with dimensions 20 cm and 15 cm. When the lid is open the ants run at a constant speed 1 cm/sec in all directions away from the place they were confined, and occupy a region around ABCD. Draw the region where the ants can be found after 5 sec after their release.*

The students realized that in order to escape further away the ants must move in the direction perpendicular to the boundary of the rectangular lid. This brings them up to 5 cm away from the boundary. Naturally, those ants that escape through the corner and travel 5 cm in all possible directions can be found in a circular region around this corner. The final picture is shown on Figure 1 (right).

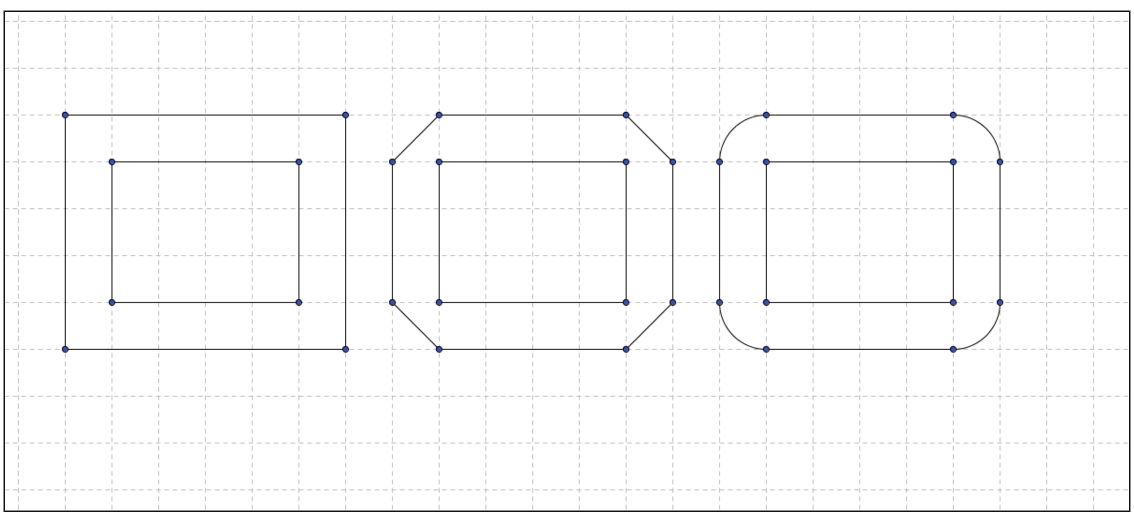

Figure 1. Various ideas about a ‘region of uniform width around a rectangle’.

From this new problem situation, the students came to a different model of a region of uniform width around a rectangle. The students were invited to discuss together how to formalize this notion and generalize it to shapes other than rectangular. They agreed that if one takes circles of fixed radius and places them so that the centers are on the perimeter of the rectangle, the union of all the circles would mark out the required exterior boundary. This approach also works for shapes other than rectangular (see figure 2). To construct the region of uniform width w around a given figure F, one takes the curvilinear band swiped out by a circle of fixed radius w as its center moves along the boundary of F and removes from it the points that belong to F.

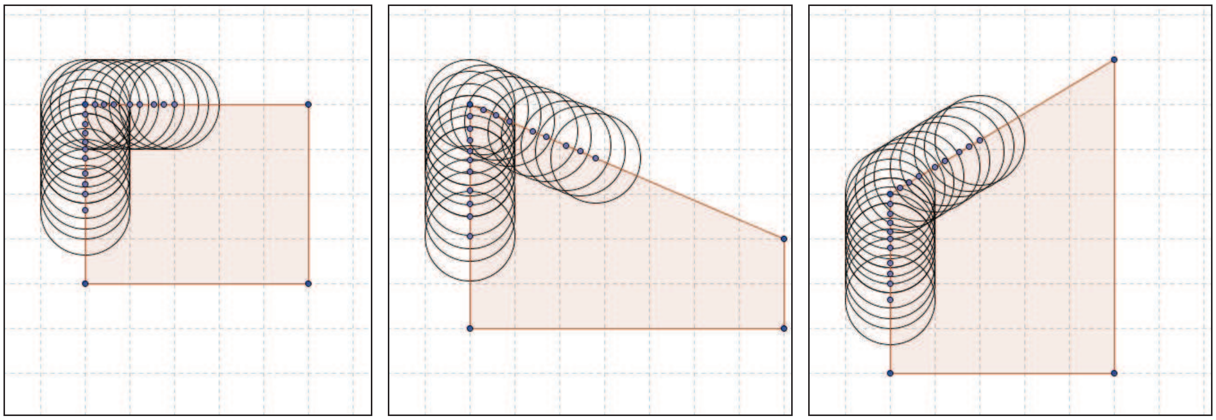

Figure 2. Construction of the region of uniform width around various shapes by placing circles on its perimeter.

It is remarkable that this approach found by the two students who were guided by a problem situation, is fully consistent with the notion of a 'uniform neigborhood of a set' (Wikipedia, 2012) as defined in higher mathematics (also known as 'ball around a set' – see e.g. Berger, Vol. 1, p.2).

I asked a few mathematics teachers about why the model of nested rectangles is so popular in grade school and why an alternative model is almost never discussed. There were several answers ranging from "the students perfectly accept the nested rectangle model" to "this type of problems lead to a quadratic equation that is easy to set up for the nested model because it deals with areas of rectangles."

The main purpose of this article is to argue that, given an opportunity, students are capable to challenge the nested rectangle model and, with an appropriate guidance, to develop an alternative model that is more consistent with higher mathematics. By letting students to consider both models, the teacher takes into account students' continuing cognitive development from primary to university level (Kondratieva, 2011).

As for setting up a quadratic equation, the second model deals with areas of rectangles and circles and thus remains within the scope of grade school programme. As we see from several examples given in this article, the quadratic equation resulting from the nested rectangle model has the form $4x^2 + px = q$, where p is the perimeter of a given rectangle, x is the unknown width and q is given area of the region of uniform width around the rectangle. The quadratic equation resulting from the model with rounded corners differs only by replacing coefficient 4 with $\pi \approx 3.14$ in front of the x^2 term. Therefore, with minimal algebraic modifications the new model enriches learners' geometrical view on the problem.

Conclusion

As noted in the Introduction, teaching of mathematics in grade school relies on many concrete examples. Mathematical notions the students learn and cognitive schemata they form come from generalization and extension of their concrete experience. In order to develop critical thinking they should be given a variety of situations and some freedom of interpretation. This article shows that examples referring to a picture frame or a swimming pool walk lead to one possible (simplified) way to introduce the notion of a region of uniform width around a rectangle. But if the question is stated differently, for instance like in the problem about running ants, a more advanced model of this notion also becomes accessible for students at this level. Development of students' mathematical reasoning will benefit from discussion of both models in mathematical classrooms in the appropriate context.

References

1. Answerbag (2008). Retrieved March 20, 2012, from http://www.answerbag.com/q_view/570776
2. Berger, M. (1987). *Geometry*. Springer Verlag.
3. Kondratieva, M. (2011). The promise of interconnecting problems for enriching students' experiences in mathematics. *The Montana Mathematics Enthusiast*, Vol. **8** Nos 1–2, 355–382.
4. Kwiznet (2012). Retrieved March 20, 2012, from http://www.kwiznet.com/p/takeQuiz.php?ChapterID=2986&CurriculumID=24
5. Mymathforum (2011). Retrieved March 20, 2012, from http://www.mymathforum.com/viewtopic.php?f=13&t=24475
6. Poincaré, H. (1996). *Science and Method*. Bristol: Thoemmes Press. (Original work published 1914)
7. Small, M. (2009). *Good Questions: Great Ways to Differentiate Mathematics Instruction.* Teachers College Press.
8. Tall, D. O., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., Cheng, Y.-H. (2012). The Cognitive Development of Proof. In G. Hanna, M. de Villiers (Eds.) *Proof and proving in Mathematics Education.* Springer. New ICMI Study Series, Vol. **15**.
9. Wikipedia (2012). Retrieved March 20, 2012, from http://en.wikipedia.org/wiki/Neighbourhood_(mathematics)