## Discrete Green's theorem,

 series convergence acceleration, and surprising identities.Margo Kondratieva<br>Memorial University of Newfoundland

A function which is a derivative of a known function can be integrated in a closed form due to the Fundamental Theorem of Calculus (FTC). Similarly, a series whose terms are differences of successive members of a known sequence can be summed by an elementary method known as telescoping.

To make our point explicit, let $f \in C[0, \infty)$ be such that

$$
f(x)=-u^{\prime}(x), \quad \text { and } \quad \lim _{x \rightarrow \infty} u(x)=0 .
$$

Then, according to the FTC applied to the improper integral,

$$
\int_{1}^{\infty} f(x) d x=u(1)
$$

Similarly, consider a sequence $f(n), n=1,2,3 \ldots$ such that

$$
f(n)=u(n)-u(n+1) \quad \text { and } \quad \lim _{n \rightarrow \infty} u(n)=0 .
$$

Then

$$
\begin{equation*}
\sum_{n=1}^{\infty} f(n)=u(1) \tag{1}
\end{equation*}
$$

## Telescoping-I (Simple examples)

Consider the folowing algebraic relations

$$
\frac{1}{z(z+1)}=\frac{1}{z}-\frac{1}{z+1}
$$

hence

$$
\sum_{z=1}^{\infty} \frac{1}{z(z+1)}=1
$$

Similarly,

$$
\frac{2}{z(z+1)(z+2)}=\frac{1}{z(z+1)}-\frac{1}{(z+1)(z+2)}
$$

hence

$$
\sum_{z=1}^{\infty} \frac{1}{z(z+1)(z+2)}=\left(\frac{1}{2}\right)\left(\frac{1}{1 \cdot 2}\right)=\frac{1}{4}
$$

In general,

$$
\begin{equation*}
\frac{m}{(z)_{m+1}}=\frac{1}{(z)_{m}}-\frac{1}{(z+1)_{m}} \tag{2}
\end{equation*}
$$

where

$$
(z)_{m}=z(z+1)(z+2) \cdot \ldots \cdot(z+m-1)
$$

For the function and its antidifference

$$
f(n)=\frac{1}{n(n+1) \cdots(n+m)}, \quad u(n)=\frac{1 / m}{n(n+1) \cdots(n+m-1)}
$$

the discrete FTC yields (since $u(1)=1 / m!$ )

$$
\sum_{n=1}^{\infty} \frac{1}{n(n+1) \cdots(n+m)}=\frac{1}{m \cdot m!}
$$

## Telescoping-II: Stirling

Various formulas can be obtained by considering linear combinations (possibly infinite) of identities (2) for different $m$ :

$$
\sum_{m=1}^{\infty} \frac{b_{m}}{(z)_{m+1}}=\sum_{m=1}^{\infty} \frac{b_{m}}{m(z)_{m}}-\frac{b_{m}}{m(z+1)_{m}}
$$

Denote

$$
f(z)=\sum_{m=1}^{\infty} \frac{b_{m}}{z(z+1) \cdots(z+m)}
$$

and

$$
u(z)=\sum_{m=1}^{\infty} \frac{b_{m} / m}{z(z+1) \cdots(z+m-1)}
$$

Then

$$
f(z)=u(z)-u(z+1)
$$

Put $z=N+n$ and observe that

$$
\sum_{n=1}^{\infty} f(N+n)=u(N+1)
$$

which results in

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_{m}}{(N+n)_{m+1}}=\sum_{m=1}^{\infty} \frac{b_{m}}{m(N+1)_{m}} . \tag{3}
\end{equation*}
$$

We'll refer to this formula as to the Stirling Reduction Formula (SRF).

Series rearrangements, which are clever elaborations of the telescoping method, go back to the pioneering works by Stirling and his contemporaries in the 18th century.


Another trick turned to a technique by Stirling is his inverse factorial formulas for $z^{n}, n \in \mathbb{Z}$. For example, repeatedly employing (2) for $m=1,2, \ldots$

$$
\begin{aligned}
\frac{1}{z} & =\frac{1}{z+1}+\frac{1}{z(z+1)} \\
& =\frac{1}{z+1}+\frac{1}{(z+1)(z+2)}+\frac{2}{z(z+1)(z+2)}
\end{aligned}
$$

.

$$
=\sum_{m=1}^{\infty} \frac{(m-1)!}{(z+1) \cdots(z+m)},
$$

He uses this formula and the SRF for series convergence acceleration.

For example, Stirling calculates $\zeta(2)$ by writing

$$
\zeta(2)=\sum_{n=1}^{N} \frac{1}{n^{2}}+\sum_{n=1}^{\infty} \frac{1}{(N+n)^{2}},
$$

summing the first $N$ terms directly and using

$$
\frac{1}{z^{2}}=\sum_{m=1}^{\infty} \frac{(m-1)!}{z(z+1) \cdots(z+m)},
$$

to transform the tail:

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{(N+n)^{2}}=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(m-1)!}{(N+n)(N+n+1) \cdots(N+n+m)} . \tag{4}
\end{equation*}
$$

The double series can be reduced (by SRF) to the form

$$
\begin{equation*}
\sum_{m=1}^{\infty} \frac{(m-1)!}{m(N+1)(N+2) \cdots(N+m)} \tag{5}
\end{equation*}
$$

From

$$
\lim _{m \rightarrow \infty} \frac{(m-1)!}{(N+1)(N+2) \cdots(N+m)} m^{N+1}=N!,
$$

one can see that the terms of series (5) decrease as $O\left(1 / m^{N+2}\right)$.
Taking $N=12$ and $K=13$ in the obtained approximate formula

$$
\zeta(2) \approx \sum_{n=1}^{N} \frac{1}{n^{2}}+\sum_{m=1}^{K} \frac{(m-1)!}{m(N+1)(N+2) \cdots(N+m)},
$$

Stirling in 1730 found $\zeta(2) \approx 1.644934065$.

Remarks. $1^{\circ}$ Quoting J. Tweedle [10]:
The problem of determining the value of $\sum n^{-2}$ exactly was at the time a celebrated problem and was first resolved by Euler in the early 1730s. In a letter of 8 June 1736 Euler communicated to Stirling the values of $\sum 1 / n^{2 k} \quad(k=1,2,3,4)$, in particular, $\sum 1 / n^{2}=\pi^{2} / 6$, and in his reply of 16 April 1738 Stirling said of Euler's result: "I acknowledge this to be quite ingenious and entirely new and I do not see that it has anything in common with the accepted methods, so that I readily believe that you have drawn it from a new source."
$2^{\circ}$ Note that formulas (2) allow one to arrive at a series which converges at a geometrical rate (not found in Stirling's book )

$$
\zeta(2)=\sum_{n=1}^{\infty} \frac{3(n-1)!^{2}}{(2 n)!}
$$

convergent geometrically with common ratio $1 / 4$. For this, one has to begin each next accelerating step (transition convergence exponent $N$ to $N+1$ ) with delay, leaving the first term of the just transformed part out.
Extending a delay to 2 and more terms leads to even faster convergent series. For example

$$
\zeta(2)=\sum_{n=0}^{\infty} \frac{46 n^{3}+104 n^{2}+77 n+19}{2\left(2 n^{2}+3 n+1\right)} \cdot \frac{n!(2 n)!}{(3 n+3)!}
$$

converges geometrically with common ratio $4 / 27$.
$3^{\circ}$ It is possible to obtain geometrically convergent series for $\zeta(2)$ with as small common ratio as desirable. Moreover, making the delay of transition from $N$ to $N+1$ increasing with $N$, one obtains faster than geometric convergence. However, it's not clear whether a general term of such a series can be written in a closed form.

## Next dimension: Green Formula

Let us return to the relation (4)=(5), or

$$
\sum_{n=1}^{\infty} \frac{1}{(N+n)^{2}}=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(m-1)!}{(N+n)_{m+1}}=\sum_{m=1}^{\infty} \frac{(m-1)!}{m \cdot(N+1)_{m}}
$$

It can be schematically rewritten as

$$
\sum_{n=1}^{\infty} F(n)=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} h(n, m)=\sum_{n=1}^{\infty} G(m)
$$

Identities of this type can be related to a two-dimensional version of FTC, i.e. Green's theorem for a conservative vector field. Let $\vec{f}=(G, F)^{T}=-\nabla u$, or equivalently,

$$
\frac{\partial F(x, y)}{\partial x}=\frac{\partial G(x, y)}{\partial y}
$$

Then line integral along a simple closed curve vanishes:

$$
\oint_{\gamma} G d x+F d y=0 .
$$

Taking a rectangular contour $\gamma$, with sides $x=0, x=L, y=0, y=L$, and letting $L \rightarrow \infty$, one obtains

$$
\int_{0}^{\infty} F(0, y) d y=\int_{0}^{\infty} G(x, 0) d x
$$

provided that

$$
\lim _{L \rightarrow \infty} \int_{0}^{\infty} G(x, L) d x=0, \quad \lim _{L \rightarrow \infty} \int_{0}^{\infty} F(L, y) d y=0
$$

We have a similar statement for series. Let there exist a pair of two-index sequences $F(n, k)$ and $G(n, k)$ such that for all $n, k \geq 1$

$$
\begin{equation*}
F(n+1, k)-F(n, k)=G(n, k+1)-G(n, k) \tag{6}
\end{equation*}
$$

and

$$
\lim _{L \rightarrow \infty} \sum_{n=0}^{\infty} G(n, L)=0, \quad \lim _{L \rightarrow \infty} \sum_{k=0}^{\infty} F(L, k)=0
$$

Then

$$
\begin{equation*}
\sum_{k=0}^{\infty} F(0, k)=\sum_{n=0}^{\infty} G(n, 0) . \tag{7}
\end{equation*}
$$

The series transformation technique based on the discrete Green formula was first proposed some 115 years ago by A. Markov, Sr., a distinguished analyst, who is the best known for his work in probability theory.


Markov explained the general approach plainly and clearly, but examples he worked out were so amazingly complicated that his method didn't find followers for more than a century.

## Samples of Markov's accelerated series

The formula

$$
\begin{equation*}
\zeta(3)=\frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2 n}{n} n^{3}} \tag{8}
\end{equation*}
$$

which is often attributed to Apéry (but also found in paper of 1953 by M. M. Hjörtnæs) is a particular case of a formula found in Markov's memoir [5] of 1890

$$
\begin{align*}
\sum_{n=0}^{\infty} & \frac{1}{(a+n)^{3}} \\
& =\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^{n} n!^{6}}{(2 n+1)!} \cdot \frac{5(n+1)^{2}+6(a-1)(n+1)+2(a-1)^{2}}{[a(a+1) \ldots(a+n)]^{4}} . \tag{9}
\end{align*}
$$

The series (8), (9) converge at the geometric rate with common ratio $1 / 4$. The following series, also known to Markov in 1890,

$$
\begin{equation*}
\zeta(3)=\frac{1}{4} \sum_{n=1}^{\infty}(-1)^{n-1} \frac{56 n^{2}-32 n+5}{(2 n-1)^{2} n^{3}} \frac{(n!)^{3}}{(3 n)!}, \tag{10}
\end{equation*}
$$

converges at the geometric rate with ratio $1 / 27$.
Ch. Hermite, the then-editor of Comptes Rendus, wrote Markov: "Par quelle voie vous êtes parvenue à une telle transformation, et il me faut vous laisser vôtre secret." ("I can't even remotely guess the way you arrived at such a transformation, and it remains to leave your secret with you.")

Having been deeply involved in studies on continued fractions throughout the 1880s, Markov had correspondence with T. J. Stieltjes (18561894) and closely watched his publications. In 1887 Stieltjes [8] published a table of the values of the Riemann Zeta function $\zeta(k)$ with 32 decimals for integral values of $k$ from 2 to 70 .


Markov might have felt challenged by that achievement and by Stieltjes' convergence acceleration technique. Possibly, it was this challenge and rivalry that prompted Markov to develop his new acceleration method. Afterwards he jealously beat Stieltjes' record, taking 22 terms in his series and obtaining the result with 33 decimals in

$$
\zeta(3)=1.202056903159594285399738161511450 .
$$

## Markov's method: general outline

Markov takes

$$
\begin{align*}
& F(n, k)=H(n, k)\left(A(n)+B(n) k+C(n) k^{2}\right),  \tag{11}\\
& G(n, k)=H(n, k)\left(\tilde{A}(n)+\tilde{B}(n) k+\tilde{C}(n) k^{2}\right), \tag{12}
\end{align*}
$$

where $H(n, k)$ is a hypergeometric function, and $A(n), B(n), C(n), .$. are yet unknown. Substitution into (6) leads to a system of first-order linear recurrence relations for $A(n), B(n), C(n), .$. with polynomial in $n$ and $k$ coefficients. The initial conditions are $A(0)=1, B(0)=0$, $C(0)=0$. If the system has a solution, then (7) gives

$$
\sum_{n=0}^{\infty} H(0, k)=\sum_{k=0}^{\infty} H(n, 0) \tilde{A}(0)
$$

and it so happens in Markov's examples that series on the right-hand side converge much faster then series on the left-hand side.

Markov also considered accelerating transformations for $q$-series.
Some authors of textbooks and monographs (Fabry, Bromwich, Knopp, Romanovski) mentioned Markov's acceleration transformation and gave relatively simple examples but it seems that no one ventured to add anything comparable to Markov's advanced examples.

Due to complexity of calculations the method remained largely ignored and was eventually forgotten. It has not been in use for almost a century.

## Gosper and computers

In 1976 William Gosper (knowing about Stirling's work but not about Markov's) gave a new life to the topic.


By the end of 1980s a powerful computer-enhanced series transformation machinery has emerged due to Wilf and Zeilberger (WZ). The WZ method proved to be very efficient in explaining known and discovering new summations formulae and in applications to problems of series acceleration and irrationality proofs.

The main feature of Gosper's algorithm [1] is so called splitting function $s_{n}$, which splits each term $a_{n}$ of a series in the proportions

$$
a_{n}=a_{n} s_{n}+a_{n}\left(1-s_{n}\right),
$$

with further recombining the left fragment of each term with right fragment of the preceding term attempting to make the combinations vanish or become very small.

## Simple examples of Gosper's transformation

1. Abel's summation-by-parts formula

$$
\sum_{n=0}^{M} a_{n}=\sum_{n=0}^{M} q_{n} \Delta p_{n}=p_{M+1} q_{M+1}-p_{0} q_{0}-\sum_{n=0}^{M} p_{n+1} \Delta q_{n}
$$

where $\Delta p_{n}=p_{n+1}-p_{n}$, is effected by the splitting function

$$
s_{n}=-\frac{p_{n}}{\Delta p_{n}}
$$

so that

$$
a_{n}\left(1-s_{n}\right)+a_{n+1} s_{n+1}=-p_{n+1} \Delta q_{n}
$$

2. Euler's formula

$$
\sum_{n=0}^{M}\left(1-a_{n+1}\right) a_{1} a_{2} \cdots a_{n}=1-a_{1} a_{2} \cdots a_{M+1}
$$

Here $s_{n}=\left(1-a_{n+1}\right)^{-1}$.
Application:

$$
\sum_{k=1}^{n} \frac{(a)_{k}^{2}}{(b)_{k}^{2}}(a+b+2 k-1)=\frac{1}{b-a-1}\left(a^{2}-\frac{(a)_{n+1}^{2}}{(b)_{n}^{2}}\right)
$$

where $(a)_{k}=a(a+1) \cdots(a+k-1)$.
(This problem was submitted by Ramanujan to the Journal of the Indian Mathematical Society, v. 7 (1915), p.199).

## Hypergeometric integrating factors

The above trivial example should not create a misleading impression: in fact, Gosper's algorithm can handle much more general situations.

Let $a(k)$ be a hypergeometric sequence, i.e, the value $a(k+1) / a(k)$ be a rational function of $k$. Then Gosper's algorithm solves a difference equation

$$
r(k+1)=\frac{a(k)}{a(k+1)}(r(k)+1)
$$

and, if successful, returns a sequence $r(k)$ such that

$$
a(k)=b(k-1)-b(k), \quad b(k)=r(k) a(k)
$$

In this case the original sequence is said to be gosperable, a term coined by Doron Zeilberger.

Zeilberger considered a more general problem:
Given $H(n, k)$ hypergeometric in both indices, find

$$
f(n)=\sum_{k} H(n, k) .
$$

If $H(n, k)$ is gosperable with respect to $k$, then $f(n)$ can be found by telescoping.

Zelberger's algorithm offers a solution to the problem even if $H(n, k)$ is not gosperable w.r.t $k$,

## Zelberger's algorithm



Given $H(n, k)$ hypergeometric in both indices, the algorithm produces sequences $a_{0}(n), a_{1}(n), \ldots a_{J}(n)$ and a two-index sequence $R(n, k)$ (the Wilf-Zeilberger Sertificate) such that

$$
G(n, k)=R(n, k) H(n, k)
$$

and the following expression is gosperable

$$
\begin{align*}
a_{0}(n) H(n, k)+\cdots & +a_{J}(n) H(n+J, k) \\
& =G(n, k+1)-G(n, k) \tag{13}
\end{align*}
$$

for some number $J$.
"The observed fact is that $99 \%$ of the time, $J=1$."
[7], p. 123.
Summation with respect to $k$ of the above relation leads to a linear recurrence relation for the unknown $f(n)$

$$
a_{0}(n) f(n)+\cdots+a_{J}(n) f(n+J)=0
$$

solvable explicitly by Petkovšec algorithm.

## Conclusion

Interestingly, Markov's approach (6), (11),(12) with

$$
B(n)=C(n)=0,
$$

is compatible with WZ (13) for $J=1$, if one puts

$$
\begin{gathered}
a_{0}(n)=-\frac{A(n)}{\Phi(n)}, \\
a_{1}(n)=\frac{A(n+1)}{\Phi(n)}, \\
R(n, k)=\frac{\tilde{A}(n)+\tilde{B}(n) k+\tilde{C}(n) k^{2}}{\Phi(n)},
\end{gathered}
$$

for an appropriate function $\Phi(n)$.
Equation (6)

$$
F(n+1, k)-F(n, k)=G(n, k+1)-G(n, k)
$$

defines the so called Wilf-Zeilberger (WZ) pair and is a cornerstone to the whole approach.

The aim of our paper [2] was to resurface the memoir by Markov and to review his old and forgotten results from the point of a modern algorithmic approach.

This resulted in amendments to Zelberger's algorithm proposed by M. Mohammed and D. Zeilberger and including a certain freedom contained in Markov's approach 11,12.

New accelerated series (or rather acceleration algorithms), in particular for $\zeta(5)$, were produced with the help of computer algebra [6].

## References

[1] R. Wm. Gosper, Jr., A calculus of series rearrangements, in: J.F. Traub (Ed.), Algorithms and complexity: New directions and recent results (Pittsburgh, Pa., 1976), Academic Press, N.-Y., 1976, pp. 121-151.
[2] M. Kondratieva and S. Sadov, Markov's transformation of series and WZ method, Adv. in Applied Math 34(2005), 393-407
[3] M. Kondratieva and S. Sadov, Summation technique forgotten for a century: Markov (1890) - Wilf-Zeilberger (1990), ICM-2002, Abstracts of Short Communications, Higher Education Press, Beijing, 2002, p. 404.
[4] A. A. Markoff, Sur les séries $\sum 1 / k^{2}, \sum 1 / k^{3}$. (Extrait d'une lettre adressée à M. Hermite, C. R. Acad. Sci. Paris 109 : 25 (1889), 934-935.
[5] A. A. Markoff, Mémoire sur la transformation des séries peu convergentes en séries très convergentes, Mém. de l'Acad. Imp. Sci. de St. Pétersbourg, t. XXXVII, No. 9 (1890), 18 pp.
[6] M. Mohammed and D. Zeilberger, The Markov-WZ Method, Elect. J. Combin. 11:1 (2004), \#R53.
[7] M. Petkovšek, H. Wilf, and D. Zeilberger, $A=B$, A. K. Peters Ltd., Natick MA, 1997.
[8] T. J. Stieltjes, Tables des valeurs des sommes $S_{k}=\sum_{n=1}^{\infty} n^{-k}$, Acta Math. 10 (1887), 299-302.
[9] J. Stirling, Methodus differentialis, London, 1730.
[10] I. Tweddle, James Stirling's Methodus Differentialis: An Annotated Translat-ion of Stirling's Text, Springer, N.-Y., 2003.

