Exercise Set 3 for Quiz on Fri Feb 1. Answers

3. Find coordinates of vector $\vec{v}$ in given basis. Is the basis orthogonal? Is the basis orthonormal?
a) $\vec{v} = (1, 2)^T$ w.r.t basis $\vec{f} = (2, 2)^T$, $\vec{g} = (3, 1)^T$.

Solution
The coordinates are $(5/4, -1/2)$.
Basis $\vec{f} = (2, 2)^T$, $\vec{g} = (3, 1)^T$ is neither orthogonal nor orthonormal.

b) $\vec{v} = (3, 5, 10)^T$ w.r.t basis $\vec{f} = (1, 2, 0)^T$, $\vec{g} = (1, 0, 2)^T$, $\vec{h} = (0, 1, 2)^T$.

Solution
The coordinates are $(1, 2, 3)$.
The basis is neither orthogonal nor orthonormal.

c) $\vec{v} = (13, -20, 15)^T$ w.r.t basis $\vec{f} = (1, -2, 3)^T$, $\vec{g} = (-1, 1, 1)^T$.

Solution
The coordinates are $(7, -6)$.
The basis is orthogonal but not orthonormal.

d) $\vec{v} = (14, 1, -8, 5)^T$ w.r.t basis $\vec{f} = (2, -1, 0, 3)^T$, $\vec{g} = (2, 1, -2, -1)^T$.

Solution
The coordinates are $(3, 4)$.
The basis is orthogonal but not orthonormal.

4. Perform (1) and (2) for each a) and b).
(1) Find matrix of the coordinate transformation for a change of basis from $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to basis $(\vec{f}_1, \vec{f}_2, \vec{f}_3)$;
(2) Let $\vec{v} = \vec{e}_1 - \vec{e}_2 + \vec{e}_3$. Find coordinates of this vector w.r.t basis $(\vec{f}_1, \vec{f}_2, \vec{f}_3)$;

if

a) $\vec{f}_1 = 3\vec{e}_1 - 5\vec{e}_2 + \vec{e}_3$, $\vec{f}_2 = 5\vec{e}_1 - 10\vec{e}_2 + 5\vec{e}_3$, $\vec{f}_3 = 2\vec{e}_1 - \vec{e}_3$.

b) $\vec{e}_1 = -2\vec{f}_1 + \vec{f}_2 + 3\vec{f}_3$, $\vec{e}_2 = -3\vec{f}_1 + \vec{f}_2 + 2\vec{f}_3$, $\vec{e}_3 = -4\vec{f}_1 + 2\vec{f}_2 + \vec{f}_3$.

Solution
a) (1)From the coefficients in formulas $\vec{f}_1 = 3\vec{e}_1 - 5\vec{e}_2 + \vec{e}_3$, $\vec{f}_2 = 5\vec{e}_1 - 10\vec{e}_2 + 5\vec{e}_3$, $\vec{f}_3 = 2\vec{e}_1 - \vec{e}_3$ we directly construct matrix of the coordinate transformation for a change of basis from $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to basis $(\vec{f}_1, \vec{f}_2, \vec{f}_3)$

$$M_{e\to f} = \begin{bmatrix} 3 & 5 & 2 \\ -5 & -10 & 0 \\ 1 & 5 & -1 \end{bmatrix}$$

(2)From $\vec{v} = \vec{e}_1 - \vec{e}_2 + \vec{e}_3$ we find coordinates of $\vec{v}$ w.r.t basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$: $X = (1, -1, 1)^T$.

Denote by Y coordinates of $\vec{v}$ w.r.t basis $(\vec{f_1},\ \vec{f_2},\ \vec{f_3})$. Then

$$X = M_{e\to f} Y.$$

Thus $Y = (M_{e\to f})^{-1} X = (-3/5; 2/5; 2/5)^T$ or $\vec{v} = -0.6\vec{f_1} + 0.4\vec{f_2} + 0.4\vec{f_3}$. (I left for you to do the calculation.)

b) (1) From given formulas $\vec{e}_1 = -2\vec{f_1} + \vec{f_2} + 3\vec{f_3}$, $\vec{e}_2 = -3\vec{f_1} + \vec{f_2} + 2\vec{f_3}$, $\vec{e}_3 = -4\vec{f_1} + 2\vec{f_2} + \vec{f_3}$, we find the matrix of the change of coordinates from basis $(\vec{f_1},\ \vec{f_2},\ \vec{f_3})$ to basis $(\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3)$

$$M_{f\to e} = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

In order to find the required matrix we need to invert this one:

$$M_{e\to f} = (M_{f\to e})^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 5 & 2 \\ -5 & -10 & 0 \\ 1 & 5 & -1 \end{bmatrix}.$$

(2) From $\vec{v} = \vec{e}_1 - \vec{e}_2 + \vec{e}_3$ we find coordinates of $\vec{v}$ w.r.t basis $(\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3)$: $X = (1, -1, 1)^T$.
Denote by Y coordinates of $\vec{v}$ w.r.t basis $(\vec{f_1},\ \vec{f_2},\ \vec{f_3})$. Then

$$X = M_{e\to f} Y,$$

or equivalently,

$$M_{f\to e} X = Y.$$

Thus $Y = (-3, 2, 2)^T$ or $\vec{v} = -3\vec{f_1} + 2\vec{f_2} + 2\vec{f_3}$.