Week 8-9. Inner product spaces. (revised version)

Section 3.1 Dot product as an inner product.

Consider a linear (vector) space V. (Let us restrict ourselves to only real spaces that is we will not deal with complex numbers and vectors.)

Definition 1. An *inner product* on V is a function which assigns a real number, denoted by $<\vec{u},\vec{v}>$ to every pair of vectors $\vec{u},\vec{v}\in V$ such that
(1) $<\vec{u},\vec{v}>=<\vec{v},\vec{u}>$ for all $\vec{u},\vec{v}\in V$;
(2) $<\vec{u}+\vec{v},\vec{w}>=<\vec{u},\vec{w}>+<\vec{v},\vec{w}>$ for all $\vec{u},\vec{v},\vec{w}\in V$;
(3) $<k\vec{u},\vec{v}>=k<\vec{u},\vec{v}>$ for any $k\in\mathbf{R}$ and $\vec{u},\vec{v}\in V$.
(4) $<\vec{v},\vec{v}>\geq 0$ for all $\vec{v}\in V$, and $<\vec{v},\vec{v}>=0$ only for $\vec{v}=\vec{0}$.

Definition 2. *Inner product space* is a vector space equipped with an inner product.

It is straightforward to check that the dot product introduces by $\vec{u}\cdot\vec{v}=\sum_{j=1}^{n}u_jv_j$ is an inner product. You are advised to verify all the properties listed in the definition, as an exercise. The dot product is also called *Euclidian inner product*.

Definition 3. *Euclidian vector space* is $\mathbf{R}^n$ equipped with Euclidian inner product $<\vec{u},\vec{v}>=\vec{u}\cdot\vec{v}$.

Definition 4. A square matrix A is called *positive definite* if $\vec{v}^TA\vec{v}>0$ for any vector $\vec{v}\neq\vec{0}$.

Problem 1. Show that $\begin{bmatrix}2&0\\0&3\end{bmatrix}$ is positive definite.
Solution: Take $\vec{v}=(x,y)^T$. Then $\vec{v}^TA\vec{v}=2x^2+3y^2>0$ for $(x,y)\neq(0,0)$.

Problem 2. Show that $\begin{bmatrix}2&-1\\-1&1\end{bmatrix}$ is positive definite.
Solution: Take $\vec{v}=(x,y)^T$. Then $\vec{v}^TA\vec{v}=2x^2-2xy+y^2=x^2+(x-y)^2>0$ for $(x,y)\neq(0,0)$.

Another definition is equivalent to the previous one but it is a better tool for checking whether or not a matrix is positive definite. The equivalence is not easy the see and it will be proved later.

Definition 5. A square matrix A is called *positive definite* if it is symmetric and all its eigenvalues are positive.

The importance of positive definite matrices becomes clear from the following theorem.

Theorem 1. If $n\times n$ matrix A is positive definite then $<\vec{u},\vec{v}>=\vec{u}^TA\vec{v}$ for $\vec{u},\vec{v}\in\mathbf{R}^n$ defines an inner product on $\mathbf{R}^n$, and every inner product on $\mathbf{R}^n$ arises in this way.

That is, every inner product on $\mathbf{R}^n$ correspond to a positive definite matrix. In particular, the dot product corresponds to the Identity matrix.

Section 3.2 The norm.

The norm of a vector in an inner product space V is introduced by analogy with the notion of the magnitude (length) of a vector in Euclidian space $\mathbf{R}^n$.

Definition 6. The *norm* of a vector $\vec{v}$ in an inner product space V is a non-negative number denoted by $||\vec{v}||$ and such that $||\vec{v}||^2 =< \vec{v}, \vec{v} >$.

Theorem 2. Properties of the norm:
(1) $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$;
(2) $||k\vec{v}|| = k||\vec{v}||$ for all $k \in \mathbf{R}$ and $\vec{v} \in V$;
(3) triangle inequality: $||\vec{v} + \vec{u}|| \leq ||\vec{v}|| + ||\vec{u}||$ for all $\vec{v}, \vec{u} \in V$;
(4*) Schwartz inequality: $< \vec{u}, \vec{v} >\leq ||\vec{u}||\,||\vec{v}||$ for all $\vec{v}, \vec{u} \in V$.

Definition 7. *Unit vector* in an inner product space V is a vector with norm equal to 1.

Definition 8. *Unit ball* in an inner product space V is the set of all unit vectors in V.

Note that in Euclidian space $\mathbf{R}^2$ unit ball contains all vectors $\vec{OA}$, where O is the origin and A lies on the circle centered at the origin and radius 1. In Euclidian space $\mathbf{R}^3$ unit ball corresponds in the same way to the sphere of radius 1.

Problem 3. Let $V = \mathbf{R}^2$ and the inner product is defined as

$$< \vec{v}, \vec{u} >= \frac{1}{4}v_1u_1 + \frac{1}{9}v_2u_2.$$

Find the unit ball in this space.

Solution: Let $\vec{v} = (x, y)^T$. Then

$$||\vec{v}||^2 =< \vec{v}, \vec{v} >= \frac{x^2}{4} + \frac{y^2}{9}.$$

The unit ball consists of all $\vec{v} = (x, y)^T$ such that

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

This curve is an ellipse. Thus the unit ball in this space contains all vectors $\vec{OA}$, where O is the origin and A lies on the ellipse centered at the origin and has vertexes at $(\pm 2, 0), (0, \pm 3)$.

Section 3.3 Orthogonality in an inner product space.

Definition 9. Vectors $\vec{u}$ and $\vec{v}$ are said to be *orthogonal* in an inner product space if $< \vec{u}, \vec{v} >= 0$.

Definition 10. A set of vectors $\vec{v}_1, ...\vec{v}_n$ is said to be *orthogonal set* in an inner product space if
(1) none of the vectors is zero vector;
(2) each pair of vectors is orthogonal: $< \vec{v}_j, \vec{v}_k >= 0$ for all $j \neq k$.

Definition 11. A set of vectors $\vec{f}_1, ...\vec{f}_n$ is said to be *orthonormal set* in an inner product space if
(1) the set is orthogonal;
(2) each vector is a unit vector: $||\vec{v}_j|| = 1$ for all $1 \leq j \leq n$.

Problem 4. Let $V = \mathbf{R}^3$ with inner product $< \vec{u}, \vec{v} >= u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2 + u_3v_3$. Show that the following vectors form an orthogonal set in this space: $\vec{f}_1 = (2, -1, 0)^T$, $\vec{f}_2 = (0, 1, 1)^T$, $\vec{f}_3 = (0, -1, 2)^T$.

Solution: Check that $< \vec{f}_1, \vec{f}_2 >=< \vec{f}_1, \vec{f}_3 >=< \vec{f}_3, \vec{f}_2 >= 0$.

Definition 12. Let $\vec{f}_1, ...\vec{f}_n$ be a set of vectors. *Gram matrix* for this set is $n \times n$ matrix whose elements are inner products of pairs of the vectors:

$$M_{ij} = < \vec{f}_i, \vec{f}_j >, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n$$

Consequently in an Euclidian space $M_{ij} = \vec{f}_i \cdot \vec{f}_j$. Given vectors $\vec{f}_1, ...\vec{f}_n$ one can compose matrix A whose columns are these vectors. Then Gram matrix can be written as $M = A^T A$ in Euclidian case. (Note that $M = A^T BA$ for an inner product space with $< \vec{u}, \vec{v} >= \vec{u}^t B \vec{v}$.)

Previously we proved that vectors $\vec{f}_1, ...\vec{f}_n$ are linearly independent if and only is if $A^T A$ is invertible. This statement remains true for $M = A^T BA$ with $B \neq I$, and highlights the importance of Gram matrix. In other words,

Theorem 3. Vectors $\vec{f}_1, ...\vec{f}_n$ are linearly independent if and only if corresponding to them Gram matrix is non-degenerate ($\det M \neq 0$).

Gram-Schmidt orthogonalization algorithm is a process of construction an orthogonal basis $\vec{e}_1, ...\vec{e}_n$ in vector space S=span($\vec{f}_1, ...\vec{f}_n$), where $\vec{f}_1, ...\vec{f}_n$ is an arbitrary basis.

This process is possible in an inner product space (as well as in an Euclidian space as its particular case). It is based on the orthogonality condition of two vectors: *their inner product equals to zero.*

Algorithm:

step 1. take $\vec{e}_1 = \vec{f}_1$

step 2. take $\vec{e}_2 = \vec{f}_2 - a\vec{e}_1$, where $a = \frac{<\vec{e}_1, \vec{f}_2>}{<\vec{e}_1, \vec{e}_1>}$

step 3. take $\vec{e}_3 = \vec{f}_3 - b_1\vec{e}_1 - b_2\vec{e}_2$, where

$$b_1 = \frac{< \vec{e}_1, \vec{f}_3 >}{< \vec{e}_1, \vec{e}_1 >}, \qquad b_2 = \frac{< \vec{e}_2, \vec{f}_3 >}{< \vec{e}_2, \vec{e}_2 >}.$$

step k. take $\vec{e}_k = \vec{f}_k - c_1\vec{e}_1 - c_2\vec{e}_2 - \cdots c_{k-1}\vec{e}_{k-1}$, where

$$c_j = \frac{< \vec{e}_j, \vec{f}_k >}{< \vec{e}_j, \vec{e}_j >}, \quad j = 1, 2, ..., k-1.$$

Stop after k=n.

Problem 5. Let $\vec{f}_1 = (1, 2, 0, 1)^T$, $\vec{f}_2 = (0, 1, 2, 1)^T$, $\vec{f}_3 = (1, 2, 1, 0)^T$.

a) is the set linearly independent?

b) is the set orthogonal (in Euclidian sense)?

c) find an orthogonal(in Euclidian sense) basis in span $\vec{f}_1, \vec{f}_2, \vec{f}_3$.

Solution: Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. To check lineal independence one can either solve AX=0 and show that $X = 0$ is the only solution OR one can consider Gram matrix $M = A^T A = \begin{bmatrix} 6 & 3 & 5 \\ 3 & 6 & 4 \\ 5 & 4 & 6 \end{bmatrix}$ and show that its determinant is not zero. (In fact $\det A = -12$.)

b) Since the Gram matrix is not diagonal, the set is not orthogonal.

c) to construct an orthogonal set we follow the algorithm:

$\vec{e}_1 = \vec{f}_1 = (1, 2, 0, 1)^T$;

$\vec{e}_2 = \vec{f}_2 - a\vec{e}_1$, where $a = 1/2$; thus $\vec{e}_2 = (-1/2,\ 0,\ 2,\ 1/2)^T$;

$\vec{e}_3 = \vec{f}_3 - b_1\vec{e}_1 - b_2\vec{e}_2$, where $b_1 = 5/6$, $b_2 = 1/3$,

thus $\vec{e}_3 = (1/3,\ 1/3,\ 1/3,\ -1)$.

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Exercise Set 5 for Quiz on Fri March 7.

1. Give a definition of:
   -Euclidian space;
   -inner product;
   -inner product space;
   -positive definite matrix;
   -Gram matrix for vectors $\vec{f}_1, ..., \vec{f}_n$;
   -norm;
   - unit ball in an inner product space;
   -orthogonal basis in an inner product space;
   -orthonormal basis in an inner product space;

2. Give an example of:
   - Euclidian space;
   - inner product;
   - inner product space;
   - positive definite matrix;
   - Gram matrix for vectors $\vec{f}_1, \vec{f}_2, \vec{f}_3$;
   - norm of a vector;
   - unit ball in an inner product space;
   -orthogonal basis in an inner product space;
   -orthonormal basis in an inner product space;

3. List properties of the norm and prove the triangle inequality for norm in an inner product space.

4. Prove Pythagorean theorem: *Let $\vec{e}_1, .., \vec{e}_n$ be an orthogonal set of vectors in an inner product space. Then*

$$||\vec{e}_1 + \cdots + \vec{e}_n||^2 = ||\vec{e}_1||^2 + \cdots + ||\vec{e}_n||^2.$$

5. Prove Expansion theorem:
   Let $\vec{v}_1, .., \vec{v}_n$ be an orthonormal set of vectors in an inner product space S. Then for any $\vec{w} \in S$

$$\vec{w} = <\vec{v}_1, \vec{w}> \vec{v}_1 + <\vec{v}_2, \vec{w}> \vec{v}_2 + \cdots <\vec{v}_n, \vec{w}> \vec{v}_n.$$

6. Prove that *every orthogonal set of vectors (each of which is a non-zero vector) is linearly independent.* (Hint: use Gram matrix.)

7. Find a symmetric matrix A s.t. $<\vec{v}, \vec{u}> = \vec{v}^T A \vec{u}$ and explain whether this define an inner product.
   a) $<\vec{v}, \vec{u}> = v_1u_1 + 2v_1u_2 + 2v_2u_1 + 5v_2u_2$;
   b) $<\vec{v}, \vec{u}> = 2v_1u_1 + v_2u_2 + v_3u_3 - v_1u_2 - v_2u_1 + v_2u_3 + v_3u_2$;

8. Apply Gram-Schmidt algorithm to construct an orthogonal(in Euclidian sense) basis in the span of $\vec{f}_1, \vec{f}_2, \vec{f}_3$.

$$\vec{f}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$