Week 2-3. Change of basis in a vector space.
Section 2.1 Linear independence and span.

Definition 1. *Linear combination* of vectors $\vec{v}_1, ..., \vec{v}_n$ is a vector of the form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n,$$

where $a_1, ...a_n$ are some real numbers. These numbers are also called weights or coefficients of the vectors in the linear combinations.

Note that some of coefficients may be equal to zero. If all coefficients are equal to zero such linear combination gives the zero vector.

Definition 2. *Span* of vectors $\vec{v}_1, ..., \vec{v}_n$ is the set of all linear combinations of these vectors:

$$\{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdot + a_n\vec{v}_n | a_1 \in \mathbf{R}, ...a_n \in \mathbf{R}\}.$$

Theorem 1. Span of vectors $\vec{v}_1, ..., \vec{v}_n$ forms a linear vector space.

You can prove this theorem by checking all three properties of linear vector space.

Definition 3. Vectors $\vec{v}_1, ..., \vec{v}_n$ are *linearly dependent* if at least one of them can be written as a linear combination of the remaining $(n-1)$-vectors.

Definition 4. Vectors $\vec{v}_1, ..., \vec{v}_n$ are *linearly independent* if none of them can be written as a linear combination of the remaining $(n-1)$-vectors.

The following theorem gives a way to check whether or not given set of vectors is linearly independent.

Theorem 2. Consider a matrix A whose columns are vectors $\vec{v}_1, ..., \vec{v}_n$, and corresponding system of linear homogeneous equations $AX = 0$. If the system has only trivial vector-solution $X = 0$ then the vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent. If the system has a parametric solution then the vectors are linearly dependent.

Problem 1. Are the following vectors linearly dependent?

$$\vec{v}_1 = (2, 0, 0)^T, \vec{v}_2 = (3, 4, 0)^T, \vec{v}_3 = (5, 6, 7)^T.$$

Solution: Set up the matrix formed from the vectors:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Corresponding system of linear homogeneous equations $AX = 0$ is

$$2x_1 + 3x_2 + 5x_3 = 0, \quad 4x_2 + 6x_3 = 0, \quad 7x_3 = 0.$$

It has only trivial solution $x_1 = x_2 = x_3 = 0$. Thus the vectors are linearly independent.

Problem 2. Are the following vectors linearly dependent?

$$\vec{v}_1 = (2,0,0)^T, \vec{v}_2 = (3,4,2)^T, \vec{v}_3 = (5,6,3)^T.$$

Solution: Set up the matrix formed from the vectors and rewrite in the REF:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1.5 & 2.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Corresponding system of linear homogeneous equations $AX = 0$ is

$$x_1 + 1.5x_2 + 2.5x_3 = 0, \quad x_2 + 1.5x_3 = 0.$$

It has parametric solution $x_3 = 2t, x_2 = -3t, x_1 = -t/2$. Thus the vectors are linearly dependent. Pick a value for the parameter, say $t = 2$. Then $x_1 = -1$, $x_2 = -6$, $x_3 = 4$.

This means that $-\vec{v}_1 - 6\vec{v}_2 + 4\vec{v}_3 = 0$, which is so indeed. In other words, we can express one of the vectors as a linear combination of other two: $\vec{v}_1 = -6\vec{v}_2 + 4\vec{v}_3$.

Problem 3. Given any set on n m-dimensional vectors $\vec{v}_1, ..., \vec{v}_n$ show that if $n > m$ then the vectors are linearly dependent.

Solution: Consider system of linear homogeneous equation $AX = 0$, where matrix A is composed from vectors $\vec{v}_1, ..., \vec{v}_n$ and has size $m \times n$.

If $n > m$ then the number of variables, i.e. components in the unknown vector $X = (x_1, ..., x_n)^T$ is greater than the number of equations, m. That means that the rank of the system is at most m, and since $m < n$, the system has a parametric solution with $n - m$ parameters. By Theorem 2, these vectors are linearly dependent.

Note that if $n \leq m$ either case is possible so you need to check such a set of vectors explicitly (solve the system $AX = 0$) for being linearly dependent or independent.

Section 2.2 Basis and dimension of a linear space.

Definition 5. Let S be a vector space. A set of vectors $\vec{v}_1, ..., \vec{v}_n$ from S is a *basis* in S if **every** vector $\vec{u}$ from S can be written as a linear combination of vectors $\vec{v}_1, ..., \vec{v}_n$ in a **unique** way:

$$\vec{u} = a_1\vec{v}_1 + \cdots a_n\vec{v}_n.$$

In such case coefficients $a_1, ...a_n$ are regarded as *coordinates of vector $\vec{u}$ in basis $\vec{v}_1, ..., \vec{v}_n$*.

There is an equivalent definition which we may use as well

Definition 6. Let S be a vector space. A set of vectors $\vec{v}_1, ..., \vec{v}_n$ is a *basis* in S if both conditions hold:

1. vectors $\vec{v}_1, ..., \vec{v}_n$ are linearly independent,
2. S is span of vectors $\vec{v}_1, ..., \vec{v}_n$.

It is important to note that there are infinitely many choices for a basis in S, but they all will contain the same number of vectors.

Definition 7. *Dimension* of a vector space S is the number of vectors in its basis.

Thus, given a system of n linearly independent vectors one can introduce n-dimensional space by considering span of these vectors. This construction justifies the importance of the two notions considered earlier: linear independence and span.

Example: Vectors $\vec{v} = (1, 2, 3, 4)^T$ and $\vec{u} = (5, 6, 7, 8)^T$ are linearly independent. Thus $S = \{t\vec{v} + s\vec{u} | s \in \mathbf{R}, t \in \mathbf{R}\}$ is a 2-dimensional vector space. (More exactly, S is a two-dimensional subspace of $\mathbf{R}^4$ because $\vec{v}, \vec{u} \in \mathbf{R}^4$.)

Section 2.3 Coordinated of a vector in a basis. Change of basis.

So far we regarded $(m \times 1)$-matrices as m-dimensional vectors, and wrote $\vec{v} = (v_1, v_2, ..., v_m)^T$. This obviously can be rewritten as

$$\vec{v} = v_1\vec{E}_1 + v_2\vec{E}_2 + \cdots + v_m\vec{E}_m,$$

where vectors $\vec{E}_1, ...\vec{E}_m$ are columns of the $m \times m$ Identity matrix:

$$\vec{E}_1 = (1, 0, ..., 0)^T, \quad \vec{E}_2 = (0, 1, 0, ..., 0)^T, \quad \vec{E}_m = (0, ..., 0, 1)^T.$$

Note that vectors $\vec{E}_1, ...\vec{E}_m$ are linearly independent. Also, any m-dimensional vector can be written as a linear combination of them (belongs to the span of $\vec{E}_1, ...\vec{E}_m$). Thus, they form a basis in m-dimensional vector space $\mathbf{R}^m$

Definition 8. The basis $\vec{E}_1, ...\vec{E}_m$ is called the **standard basis** in $\mathbf{R}^m$.

Thus, so far we referred to m-dimensional vectors $\vec{v} = (v_1, v_2, ..., v_m)^T$ in terms of their coordinates in the standard basis $\vec{E}_1, ...\vec{E}_m$ (see Definition 5).

From now on we must take into account the fact that the standard basis in not the only basis in $\mathbf{R}^m$ (there are infinitely many others), and will specify which basis is considered when talk about coordinates of a vector.

Theorem 3. Consider a non-degenerate $n \times n$-matrix A ($\det A \neq 0$). The columns of A form a basis in $\mathbf{R}^n$.

Proof: Consider vector $\vec{v} = b_1\vec{E}_1 + \cdots b_n\vec{E}_n$. In other words, this vector is given by coordinates $B = (b_1, ..., b_n)^T$ in the standard basis.

Since $\det A \neq 0$, the system $AX = B$ has a unique solution for any B. This solution X gives coordinates of vector $\vec{v}$ in the basis formed from columns A_i, $i = 1, 2, ..n$ of matrix A:

$$B = x_1A_1 + \cdots x_nA_n.$$

Since is it possible to find a unique X for any B, the columns of A form a basis.

Note that any system on linear equations $AX = B$, where $\det A \neq 0$ can be interpreted as re-expansion of a vector given by its coordinated in the standard basis $\{E_j\}_{j=1}^n$ in the new basis $\{A_j\}_{j=1}^n$ formed by the columns of matrix A

$$\sum_{j=1}^n b_j\vec{E}_j = \sum_{j=1}^n x_j\vec{A}_j.$$

The equal sign meant that this is the same vector, but it has coordinates B in the standard basis and it has coordinates X in the new basis $\{A_j\}_{j=1}^n$.

The matrix A plays dual role:

1. its columns define the new basis, and

2. it help to find the coordinates of the vector in the new basis it they are known in the standard basis: $X = A^{-1}B$.

Definition 9. Let an arbitrary vector $\vec{w} \in S$ be expanded in two ways with respect to basis $\{\vec{v}_j\}$ and another basis $\{\vec{u}_j\}$ in S: $\vec{w} = x_1\vec{v}_1 + \cdots x_n\vec{v}_n = y_1\vec{u}_1 + \cdots y_n\vec{u}_n$. Let $X = (x_1, ..., x_n)^T$, $Y = (y_1, ..., y_n)^T$. A $n \times n$ non-degenerate matrix M such that $MX = Y$ for any $\vec{w} \in S$ is called a *matrix of coordinate transformation* from the basis $\{\vec{u}_j\}$ to the basis $\{\vec{v}_j\}$.

For $AX = B$, we can say that any non-degenerate $n \times n$-matrix A is the matrix of coordinate transformation from the standard basis $\{E_j\}_{j=1}^n$ to the new basis formed by its columns $\{A_j\}_{j=1}^n$.

Problem 4. Consider vector $\vec{u}$ given by coordinates $(2, 6)$ in the standard basis. Find coordinates of $\vec{u}$ in new basis given by $\vec{v}_1 = (1, 2)^T$ and $\vec{v}_2 = (3, 4)^T$.

Solution: Note that $\vec{u}$ given by coordinates $(2, 6)$ in the standard basis, that it

$$\vec{u} = 2\vec{E}_1 + 6\vec{E}_2.$$

We need to find x_1, x_2 such that

$$\vec{u} = x_1\vec{v}_1 + x_2\vec{v}_2.$$

That is equivalent to solving the system $AX = B$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Solution of this system is $x_1 = 5, x_2 = -1$. Thus

$$\vec{u} = 2\vec{E}_1 + 6\vec{E}_2 = 5\vec{v}_1 - \vec{v}_2.$$

Answer: Vector $\vec{u}$ has coordinates $(5, -1)$ in new basis $\vec{v}_1, \vec{v}_2$.

Matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is the matrix of coordinate transformation from the standard basis $\vec{E}_1 = (1, 0)^T, \vec{E}_2 = (0, 1)^T$ to new basis $\vec{v}_1 = (1, 2)^T$, $\vec{v}_2 = (3, 4)^T$.

One can consider coordinate transformation from any old basis to any new basis in a linear space (not only from the standard basis to a new one). This can be done as follows.

Let A and C be two non degenerate $n \times n$ matrices. Then each of them is a matrix of coordinate transformation from the standard basis to the new one. This new basis is $\{A_j\}_{j=1}^n$ in case of A, or $\{C_j\}_{j=1}^n$ in case of C. This means that the same vector $\vec{u}$, given by B in the standard basis, can be re-expanded with respect to each of the new basis:

$$\sum_{j=1}^{n} b_j\vec{E}_j = \sum_{j=1}^{n} x_j\vec{A}_j = \sum_{j=1}^{n} y_j\vec{C}_j.$$

Here the $X = (x_1, ..., x_n)^T$ is a solution of the system $AX = B$ and $Y = (y_1, ..., y_n)^T$ is a solution of the system $CY = B$. Thus we write

$$B = AX = CY,$$

or, equivalently, $AX = CY$, from which we have

$$MX = Y, \qquad M = C^{-1}A.$$

Here X represent coordinates of vector $\vec{u}$ in basis $\{A_j\}_{j=1}^n$, Y represent coordinates of vector $\vec{u}$ in basis $\{C_j\}_{j=1}^n$, and matrix M is the matrix of coordinate transformation from old basis $\{C_j\}_{j=1}^n$ to new basis $\{A_j\}_{j=1}^n$.

We could alternatively write

$$X = LY, \qquad L = A^{-1}C.$$

In this case matrix L is the matrix of coordinate transformation from old basis $\{A_j\}_{j=1}^n$ to new basis $\{C_j\}_{j=1}^n$. Note that $L = M^{-1}$.

Problem 5. Let $\vec{v} = (2,2)$. Consider basis $\vec{A}_1 = (2,0)^T$, $\vec{A}_2 = (0,3)^T$, and another basis $\vec{C}_1 = (1,1)^T$, $\vec{C}_2 = (-1,1)^T$. (All coordinates are given with respect to the standard basis).

a) Find coordinates of vector $\vec{v}$ in each basis.

b) Find the matrix of coordinate transformation from one basis to another.

Solution: a) Coordinates X of $\vec{v}$ in $\vec{A}_1, \vec{A}_2$ are

$$X = A^{-1}B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}.$$

Coordinates Y of $\vec{v}$ in $\vec{C}_1, \vec{C}_2$ are

$$Y = C^{-1}B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

b) Matrix of coordinate transformation from old basis $\{A_j\}_{j=1}^2$ to new basis $\{C_j\}_{j=1}^2$ is

$$A^{-1}C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/3 & 1/3 \end{bmatrix}.$$

Matrix of coordinate transformation from old basis $\{C_j\}_{j=1}^2$ to new basis $\{A_j\}_{j=1}^2$ is

$$M = C^{-1}A = \begin{bmatrix} 1 & 3/2 \\ -1 & 3/2 \end{bmatrix}.$$

The columns of the matrix of coordinate transformation also have the following meaning.

Theorem 4. Consider two non-degenerate $n \times n$ matrices A and C. Let columns $\{A_j\}_{j=1}^n$ of matrix A represent coordinates of vectors of a **new basis** in the standard basis $\{E_j\}_{j=1}^n$, while columns $\{C_j\}_{j=1}^n$ of matrix C represent coordinates of vectors of an **old basis** in the standard basis $\{E_j\}_{j=1}^n$. Then the columns of the *matrix of coordinate transformation* $M = C^{-1}A$ from the old basis to new basis represent coordinates of vectors of the new basis in the old basis.

Proof. The prove follows from the relation for matrices $CM = A$.

We give it in details for $n = 3$. Rewrite $CM = A$ as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This implies for the first column of matrix A:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} m_{11} + \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} m_{21} + \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} m_{31} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

This also true for other columns of A ($k = 1, 2, 3$)

$$\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} m_{1k} + \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} m_{2k} + \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} m_{3k} = \begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix}.$$

In the vector notations we have

$$\vec{C}_1 m_{1k} + \vec{C}_2 m_{2k} + \vec{C}_3 m_{3k} = \vec{A}_k, \qquad k = 1, 2, 3.$$

But this means precisely that the k-th column $\begin{bmatrix} m_{1k} \\ m_{2k} \\ m_{3k} \end{bmatrix}$ of matrix M gives coordinates of vector $\vec{A}_k$ in the basis $\vec{C}_1, \vec{C}_2, \vec{C}_3$, which is stated in the theorem. □

Problem 6.

a) Find matrix of the coordinate transformation for a change of basis from ($\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$) to basis ($\vec{f}_1$, $\vec{f}_2$, $\vec{f}_3$) if $\vec{f}_1 = 2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3$, $\vec{f}_2 = -5\vec{e}_2 - 6\vec{e}_3$, $\vec{f}_3 = -\vec{e}_2 + \vec{e}_3$.

b) find coordinates of vector $\vec{v}$ in old basis (e) if coordinates of this vector in the new basis (f) are $X = (1, 2, 3)^T$.

Solution: a) Use theorem 4 to find columns of the matrix of coordinate transformation

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -5 & -1 \\ 4 & -6 & 1 \end{bmatrix}.$$

b) We have $\vec{v} = \vec{f}_1 + 2\vec{f}_2 + 3\vec{f}_3$. Substitute expressions for vectors $\vec{f}_k$ in terms of vectors $\vec{e}_k$ we obtain

$$\vec{v} = (2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3) + 2(-5\vec{e}_2 - 6\vec{e}_3) + 3(-\vec{e}_2 + \vec{e}_3) = 2\vec{e}_1 - 10\vec{e}_2 - 5\vec{e}_3.$$

Thus the answer is $Y = (2, -10, -5)^T$. Note that the same answer can be obtained by the matrix multiplication:$Y = MX$, where M was found in part (a).

Note that M is invertible, and its inverse M^{-1} is the matrix of coordinate transformations from the basis ($\{\vec{f}_j\}_{j=1}^3$) to basis ($\{\vec{e}_j\}_{j=1}^3$). Thus it is useful in two ways:

1. the columns of M^{-1} give coordinates of vectors $\{\vec{e}_j\}_{j=1}^3$ in basis $\{\vec{f}_j\}_{j=1}^3$.

2. the formula $X = M^{-1}Y$ gives coordinates X of a vector in basis $\{\vec{f}_j\}_{j=1}^3$ if its coordinated Y in basis $\{\vec{e}_j\}_{j=1}^3$ are given.

Section 2.4 Orthogonal basis. Vector expansion in an orthonormal basis

Definition 10. An *orthogonal basis* $\vec{v}_1, ...\vec{v}_n$ is a basis such that any two vectors $\vec{v}_i$ and $\vec{v}_j$, $i \neq j$ are orthogonal to each other.

Definition 11. An *orthonormal basis* is an orthogonal basis such that each vector has unit length.

Theorem 5. Let $\vec{v}_1, ...\vec{v}_n$ be an orthonormal basis. Let $\vec{v}$ be a vector from the span of $\vec{v}_1, ...\vec{v}_n$.

Let in the standard basis $\{E_j\}_{j=1}^n$, the basic vector $\vec{v}_j$ has coordinates $A_j = (a_{1j}, ..., a_{nj})^T$, $j = 1, 2, ..., n$ and vector $\vec{v}$ has coordinates $B = (b_1, ..., b_n)^T$. Then

$$B = (B \cdot A_1)A_1 + (B \cdot A_2)A_2 + \cdots + (B \cdot A_n)A_n.$$

Proof: Write $B = x_1A_1 + \cdots + x_nA_n$ or $B = AX$. Then $X = A^{-1}B$. If basis is orthonormal then $A_i \cdot A_j = 0$ for $i \neq j$ and $A_i \cdot A_i = 1$. Thus $A^{-1} = A^T$, and $X = A^TB$. Therefore, $x_i = A_i \cdot B$. □

Exercise Set 3 for Quiz on Fri Feb 1.

(Note that some questions from Exercise Set 2 may appear on this quiz as well.)

1. Give a definition of:
   -standard basis;
   -orthogonal basis;
   -orthonormal basis;
   -coordinates of a vector in given basis;
   -matrix of the coordinate transformation (change of basis).

2. Give an example of:
   -orthogonal basis of a 3-dimensional vector space;
   -standard basis in 4-dimensional space;
   -finding coordinates of a vector in a given basis;
   -finding coordinates of a vector in a given orthogonal basis;
   -matrix of the coordinate transformation for a change of basis in a 2-dimensional space.

3. Find coordinates of vector $\vec{v}$ in given basis. Is the basis orthogonal? Is the basis orthonormal?

   a) $\vec{v} = (1,2)^T$ in basis $\vec{f} = (2,2)^T$, $\vec{g} = (3,1)^T$.

   b) $\vec{v} = (3,5,10)^T$ in basis $\vec{f} = (1,2,0)^T$, $\vec{g} = (1,0,2)^T$, $\vec{h} = (0,1,2)^T$.

   c) $\vec{v} = (13,-20,15)^T$ in basis $\vec{f} = (1,-2,3)^T$, $\vec{g} = (-1,1,1)^T$.

   d) $\vec{v} = (14,1,-8,5)^T$ in basis $\vec{f} = (2,-1,0,3)^T$, $\vec{g} = (2,1,-2,-1)^T$.

4. Perform (1) and (2) for each a) and b).

   (1) Find matrix of the coordinate transformation for a change of basis from $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to basis $(\vec{f}_1, \vec{f}_2, \vec{f}_3)$;

   (2) Let $\vec{v} = \vec{e}_1 - \vec{e}_2 + \vec{e}_3$. Find coordinates of this vector in basis $\vec{f}_j$, $j = 1,2,3$;

   if

   a) $\vec{f}_1 = 3\vec{e}_1 - 5\vec{e}_2 + \vec{e}_3$, $\vec{f}_2 = 5\vec{e}_1 - 10\vec{e}_2 + 5\vec{e}_3$, $\vec{f}_3 = 2\vec{e}_1 - \vec{e}_3$.

   b) $\vec{e}_1 = -2\vec{f}_1 + \vec{f}_2 + 3\vec{f}_3$, $\vec{e}_2 = -3\vec{f}_1 + \vec{f}_2 + 2\vec{f}_3$, $\vec{e}_3 = -4\vec{f}_1 + 2\vec{f}_2 + \vec{f}_3$.