Week 1 Linear vector spaces and subspaces.

Section 1.1 The notion of a linear vector space.

For the purpose of these notes we regard $(m \times 1)$-matrices as m-dimensional vectors, and write $\vec{v} = (v_1, v_2, ..., v_m)^T$. (We write standard column vectors as transposed row vectors in order to save space.)

For instance, the collection of all 2-dimensional vectors $\vec{v} = (x, y)^T$ constitutes the Euclidian plane $\mathbf{R}^2$. This collection has the properties:

(1) zero vector $(0, 0)^T$ belongs to $\mathbf{R}^2$;

(2) the sum of any two 2-dimensional vectors is again a 2-dimensional vector;

(3) a multiple of any 2-dimensional vector is again a 2-dimensional vector.

For example, $(1, 2)^T + (3, 4)^T = (4, 6)^T$ and $(-3)(1, 2)^T = (-3, -6)^T$.

The fact that $\mathbf{R}^2$ has the properties listed above makes it a linear vector space.

Similarly $\mathbf{R}^3$, a collection of 3-dimensional vectors $\vec{v} = (x, y, z)^T$ is a linear vector space because all three properties hold for it:

(1) zero vector $(0, 0, 0)^T$ belongs to $\mathbf{R}^3$;

(2) the sum of any two 3-dimensional vectors is again a 3-dimensional vector;

(3) a multiple of any 3-dimensional vector is again a 3-dimensional vector.

With these examples in mind we now give a general formal definition.

Definition 1. A *linear vector space* is a collection of vectors with the following properties:

(1) it contains the zero vector $\vec{0}$ — such that for any vector $\vec{v}$ from the collection $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$;

(2) the sum of any two vectors from the collection is again in the collection;

(3) a multiple of any vector from the collection is again in the collection.

Problem 1. Let $\vec{n} = (1, 2, 3)^T$. Consider all vectors $\vec{v} = (x, y, z)^T$ which are orthogonal to vector $\vec{n}$: $\vec{v} \cdot \vec{n} = 0$, or equivalently, $x + 2y + 3z = 0$. Show that the collection of all such vectors $\vec{v}$ is a linear vector space.

Solution. We need to check the three properties listed in the definition of linear vector space.

(1) If $x = y = z = 0$ then $x + 2y + 3z = 0$. Thus zero $(0, 0, 0)^T$ vector belongs to the collection.

(2) Let $\vec{v} = (x_1, y_1, z_1)^T$ and $\vec{u} = (x_2, y_2, z_2)^T$ are in the collection. This means $x_1 + 2y_1 + 3z_1 = 0$ and $x_2 + 2y_2 + 3z_2 = 0$.

For the sum $\vec{v} + \vec{u}$ we have:

$(x_1 + x_2) + 2(y_1 + y_2) + 3(z_1 + z_2) = (x_1 + 2y_1 + 3z_1) + (x_2 + 2y_2 + 3z_2) = 0.$

Thus, the sum of any two vectors from the collection also belongs to the collection.

(3)Let $\vec{v} = (x_1, y_1, z_1)^T$. This means $x_1 + 2y_1 + 3z_1 = 0$. For a multiple of $\vec{v}$ we have $k\vec{v} = (kx_1, ky_1, kz_1)^T$ and $kx_1 + 2ky_1 + 3kz_1 = k(x_1 + 2y_1 + 3z_1) = 0$. Thus a multiple of any vector from the collection is again in the collection.

Since all three properties hold, the collection of vectors orthogonal to the vector $\vec{n} = (1, 2, 3)^T$ is a linear vector space.

Similarly, one can prove the following statement (do it as an exercise!).

Theorem 1. Given any nonzero vector $\vec{n} = (n_1, n_2, n_3)^T$, a collection of all vectors orthogonal to $\vec{n}$ forms a linear vector space.

Remark 1. Note that geometrically this collection of vectors is a plane with normal vector $\vec{n} = (n_1, n_2, n_3)^T$ and passing through the origin. The plane has equation $n_1x + n_2y + n_3z = 0$.

For instance, if $\vec{n} = (0, 0, 1)^T$ the plane has equation $z = 0$ and consists of vectors $\vec{v} = (x, y, 0)$. This plane coincides with the Euclidian plane $\mathbf{R}^2$. In such a case we say that $\mathbf{R}^2$ is a linear subspace of $\mathbf{R}^3$.

Definition 2. A *linear subspace* of a linear vector space is any subset of this linear vector space such that it is a linear vector space itself.

Section 1.2 Geometry of linear subspaces in $\mathbf{R}^3$.

From Theorem 1 and Remark 1 it follows that:

Theorem 2. Any plane passing through the origin is a linear subspace in the linear space $\mathbf{R}^3$.

Problem 2. Show that all $(x, y, z)^T$ such that $5x - 6y + 7z = 0$ form a linear space which is a linear subspace of $\mathbf{R}^3$.

Solution. Equation $5x - 6y + 7z = 0$ describes a plane passing through the origin and having normal vector $\vec{n} = (5, -6, 7)^T$. All vectors $(x, y, z)^T$ such that $5x - 6y + 7z = 0$ belong to this plane and are orthogonal to $\vec{n} = (5, -6, 7)^T$. They form a linear vector space by Th. 1 and this space is a subspace of $\mathbf{R}^3$ by Th. 2.

Problem 3. Show that all $(x, y, z)^T$ such that $5x - 6y + 7z = 1$ does NOT form a linear subspace of $\mathbf{R}^3$.

Solution. Equation $5x - 6y + 7z = 1$ again describes a plane with normal vector $\vec{n} = (5, -6, 7)^T$. But now the plane does NOT pass through the origin because if $x = y = z = 0$ then $5x - 6y + 7z = 0 \neq 1$. This means that the zero vector does NOT belong to this collection of vectors, which by Def. 1 makes this collection NOT a linear vector space. Thus, by Def. 2 this is NOT a linear subspace of $\mathbf{R}^3$.

Problem 4. Let $\vec{d} = (1, 2, 3)^T$. Show that the collection of all vectors proportional to $\vec{d}$, that is $(x, y, z)^T = k\vec{d}$, where k is any number, forms a linear subspace in $\mathbf{R}^3$.

Solution. We have to show that this collection forms a linear vector space. Then, by Def 2, we will obtain the required statement.

In order to show that this collection forms a linear vector space we need to check all the properties in Def. 1.

(1) Zero vector belongs to the collection: if $k = 0$ then $k\vec{d} = (0,0,0)^T$.

(2) If $\vec{v}$ and $\vec{u}$ belong to the collection, that is $\vec{v} = k_1\vec{d}$ and $\vec{u} = k_2\vec{d}$, then $\vec{v}+\vec{u} = k_1\vec{d}+k_2\vec{d} = (k_1+k_2)\vec{d} = k\vec{d}$. Thus the sum also belongs to the collection.

(3) If $\vec{v}$ belong to the collection, that is $\vec{v} = k_1\vec{d}$, then $s\vec{v} = s(k_1\vec{d}) = k\vec{d}$. Thus the multiple of $\vec{v}$ also belongs to the collection.

Since all three properties hold, the collection of vectors proportional to the vector $\vec{d} = (1,2,3)^T$ is a linear vector space by Def 1.

Since the collection of vectors proportional to the vector $\vec{d} = (1,2,3)^T$ is a subset of all 3-dimensional vectors $(x,y,z)^T$ and itself forms a linear vector space, this collections is a linear subspace of $\mathbf{R}^3$.

Similarly, one can prove the following statement (do it as an exercise!).

Theorem 3. Given any nonzero vector $\vec{d} = (d_1,d_2,d_3)^T$, a collection of all vectors proportional to $\vec{d}$ forms a linear vector space. This collection is a linear subspace of $\mathbf{R}^3$.

Remark 2. Note that geometrically this collection of vectors is a line with direction vector $\vec{d} = (d_1,d_2,d_3)^T$ and passing through the origin. The line has equation $(x,y,z)^T = s(d_1,d_2,d_3)^T$, where s is any number.

From Theorem 3 and Remark 2 it follows that:

Theorem 4. Any line passing through the origin is linear space, and thus is a linear subspace in the linear space $\mathbf{R}^3$.

Problem 5. Show that the following collections of vectors are NOT linear spaces:

(a) all triples $(x,y,z)^T = (k+4, 2k+5, 3k+6)^T$, where k is any number;

(b) all triples (x,y,z) such that $x^2+y^2+z^2 = 1$;

(c) all triples (x,y,z) such that $x^2-y^2 = 0$ and $z = 0$;

(d) all triples (x,y,z) such that $x \geq 0$, $y \geq 0$, $z \geq 0$.

Solutions:

(a) This collection of vectors does not contain the zero vector $(0,0,0)$. In order to have $x = 0$ one needs to take $k = -4$, but this value of k makes $y = -3$, $z = -6$. Thus it is impossible to make all three components equal to zero with the same value of k.

Note also that collection of points $(x,y,z)^T = (k+4, 2k+5, 3k+6)^T$, where k is any number forms a line not passing through the origin.

(b) This collection of vectors does not contain the zero vector $(0,0,0)$. Let $x = y = z = 0$. Then $x^2+y^2+z^2 = 0 \neq 1$.

Note also that this collection of points forms a surface of the sphere of radius 1 with center at the origin.

(c) This collection of vectors contains the zero vector $(0, 0, 0)$: If $x = y = z = 0$ then $x^2 - y^2 = 0$.

But the sum of two vectors from the collection does not always belong to the collection. Take for example $\vec{u} = (1, 1, 0)$ and $\vec{v} = (1, -1, 0)$. Then $\vec{u} + \vec{u} = (2, 0, 0)$ does not satisfy the equation $x^2 - y^2 = 4 \neq 0$.

Note also that this collection of points forms a two lines intersecting at the origin.

(d) This collection of vectors violates 3rd property of a linear space: a multiple of any vector from the collection is not always in the collection. Take $\vec{u} = (1, 1, 1)$ and $k = -2$. Then $k\vec{u} = (-2, -2, -2)$ does not satisfy the restriction with defines the collection.

Note also that this collection of points forms the first octant of $\mathbf{R}^3$.

Next theorem is the main statement in this section because it geometrically describes all possible linear subspaces in $\mathbf{R}^3$.

Theorem 5. The only linear subspaces in $\mathbf{R}^3$ are

(1) a plane passing through the origin;

(2) a line passing through the origin;

(3) the origin itself

(4) the entire $\mathbf{R}^3$.

Remark 3. In $\mathbf{R}^3$ a line and a plane are called *proper* subspaces. The origin and the entire $\mathbf{R}^3$ are referred to as either trivial, extreme or degenerate cases.

Section 1.3 Homogeneous systems of linear equations and linear subspaces in $\mathbf{R}^3$.

In this section we consider two examples familiar from Linear Algebra (M2050) and interpret the sets of solutions as linear spaces.

Problem 6. Let A be 3×3 matrix. Show that the collection of all solutions of a homogeneous system $AX = 0$ forms a linear subspace of $\mathbf{R}^3$.

Solution: First note that a homogeneous system always has trivial solution $X = (0, 0, 0)$. Thus the collection always contains the origin.

Now we will consider different cases:

(Recall that the rank of a matrix A, denoted $\operatorname{rk} A$, is the number of the leading 1s in the row-echelon form.)

(a) Let $\operatorname{rk} A = 3$. Then there is only zero solution $X = (0, 0, 0)$, which is an example of a linear subspace, the origin by itself. An example of such a system is $x + y + z = 0$, $y + z = 0$, $z = 0$. Clearly, $x = y = z = 0$.

(b) Let $\operatorname{rk} A = 2$. Then there exist a parametric solution with one parameter. Geometrically it represents a line passing through the origin. This is another example of a linear subspace of $\mathbf{R}^3$.

An example of such a system is $x + y + z = 0$, $y + z = 0$. Clearly, $x = 0$, $y = t$, $z = -t$, where t is any number.

(c) Let $\operatorname{rk} A = 1$. Then there exist a parametric solution with two parameters. Geometrically it represent a plane passing through the origin. (We will clarify this particular representation later). This is another example of a linear subspace of $\mathbf{R}^3$.

An example of such a system is $x + y + z = 0$. Clearly, $x = -t - s$, $y = s$, $z = t$, where s, t are any numbers.

(d) Let A be a zero matrix. Then the system $AX = 0$ does not impose any restrictions on X. This gives the entire $\mathbf{R}^3$ for X.

Definition 3. A *null space* of a $n \times m$ matrix A is a collection of all $m \times 1$ vector solutions of a corresponding homogeneous system $AX = 0$.

Theorem 6. The null space of any matrix is a linear space.

This theorem is a natural generalization of our result in Problem 6. We now turn our attention to another important example.

Problem 7. Let A be a 3×3 matrix with an eigenvalue λ. Consider a collection of ALL vectors X such that $AX = \lambda X$. (Note that we allow X to be a zero vector, thus we take all eigenvectors corresponding to λ as well as the zero vector $X = (0, 0, 0)$.)

Show that this collection forms a linear subspace in $\mathbf{R}^3$.

Solution: Rewrite the relation $AX = \lambda X$ in the form $(A - \lambda I)X = 0$ and recall that λ is found from the condition $\det(A - \lambda I) = 0$. Thus, X is a solution of a homogeneous system with the matrix of coefficients $(A - \lambda I)$ of rank either 2, or 1 or 0. Referring to the previous problem, we get a parametric solution with at least one parameter. Thus we will get either a line passing through the origin or a plane through the origin, or the entire $\mathbf{R}^3$. In either case it will be a linear subspace of $\mathbf{R}^3$.

Note that we can be a little bit more precise. If the multiplicity of the eigenvalue is 1 then a line passing through the origin will be the case. If the multiplicity of the eigenvalue is 2 then either a line passing through the origin or a plane passing through the origin are possible. If the multiplicity of the eigenvalue is 3 then any of the three cases is possible.

Section 1.4 Exercises.

1. Give a definition of: linear vector space, linear subspace, null space of a matrix.
2. Give an example of: linear vector space, linear subspace, null space of a matrix.
3. Explain using the definition whether the following is a linear subspace of $\mathbf{R}^3$:
   a) any line;
   b) any two lines intersecting at the origin;
   c) the plane $x - y = 0$
   d) the plane $x - y - 2 = 0$
4. Outline the proofs of Theorems 1 and 3.
5. Give examples of matrices for each case (line, plane, entire space) in Prob.7.