1. A matrix $X$ is symmetric if $X^T = X$. Since $A$ is symmetric, $A^T = A$. Remembering that $(XY)^T = Y^TX^T$ (transpose behaves like inverse), we find $P^TAP$ is symmetric because $(P^TAP)^T = P^TA^T(P^T)^T = P^TAP$.

2. We bring $A$ to upper triangular form using only the third elementary row operation.
$$A \to \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = U'.$$
We have $U' = DU$ with
$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$
Since $A$ is symmetric, we have $A = LDU$ with $L = U^T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}$.

3. (a) $\left[\begin{array}{ccc|ccc} 0 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \end{array}\right]$
$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & -6 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{array}\right]$.
There is no inverse.

(b) $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$
$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right]$
The inverse is $\begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$

4. We have $B = A^{-1}C$ and $A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$, so
$$B = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -4 & -2 \end{bmatrix}.$$

 

5. (a) You compute $XY$ or $YX$. (Since $X$ and $Y$ are square, it is not necessary to compute both products.) If $XY = I$ or $YX = I$, then $X$ and $Y$ are inverses.

   (b) $(I - A)(I + A + A^2) = I + A + A^2 - A - A^2 - A^3 = I$, so, by (a), the inverse of $I - A$ is $I + A + A^2$.

   (c) We are asked to find the inverse of $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. Notice that $B = I - A$, where $A = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. Since $A^3 = 0$,

$$\begin{aligned} B^{-1} &= I + A + A^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$