1. Gaussian elimination on the augmented matrix gives

$$\left[\begin{array}{rrr|c} 1 & 1 & -1 & a \\ 2 & -3 & 5 & b \\ 5 & 0 & 2 & c \end{array}\right] \to \left[\begin{array}{rrr|c} 1 & 1 & -1 & a \\ 0 & -5 & 7 & b-2a \\ 0 & -5 & 7 & c-7a \end{array}\right]$$

$$\to \left[\begin{array}{rrr|c} 1 & 1 & -1 & a \\ 0 & -5 & 7 & b-2a \\ 0 & 0 & 0 & (c-7a)-(b-2a) \end{array}\right] = \left[\begin{array}{rrr|c} 1 & 1 & -1 & a \\ 0 & -5 & 7 & b-2a \\ 0 & 0 & 0 & -5a-b+c \end{array}\right]$$

There is no solution unless $-5a-b+c=0$. On the other hand, if $-5a-b+c=0$, row echelon form is

$$\to \left[\begin{array}{rrr|c} 1 & 1 & -1 & a \\ 0 & 1 & -\frac{7}{5} & -\frac{b-2a}{5} \\ 0 & 0 & 0 & 0 \end{array}\right],$$

z is a free variable, and there are infinitely many solutions.

2. $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 + c_4\mathsf{v}_4 = \mathsf{0}$ is $A\mathsf{c} = \mathsf{0}$ with $A = \begin{bmatrix} 1 & -3 & 4 & 1 \\ 2 & 0 & 2 & 1 \\ 3 & 1 & 2 & 13 \end{bmatrix}$ and $\mathsf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$.

Gaussian elimination proceeds

$$A \to \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 6 & -6 & -1 \\ 0 & 10 & -10 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since c_3 is a free variable, this homogeneous system has nontrivial solutions. For example, $-\mathsf{v}_1 + \mathsf{v}_2 + \mathsf{v}_3 + 0\mathsf{v}_4 = \mathsf{0}$. The vectors are linearly dependent. One nontrivial linear dependence equation is $\mathsf{v}_1 - \mathsf{v}_2 - \mathsf{v}_3 = \mathsf{0}$.

3. EA was formed from the operation $R2 \to R2 - 4R1$, so $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4. A is elementary, namely, that elementary matrix which effects the operation $R3 \to R3 + 5R2$. Its inverse is the elementary matrix which "undoes" A; it's the one that effects the operation $R3 \to R3 - 5R2$: $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$.

5. We bring A to upper triangular form using only the third elementary row operation.

$$A \to \begin{bmatrix} -2 & 4 & 6 \\ 0 & 4 & -5 \\ 0 & 8 & 7 \end{bmatrix} \to \begin{bmatrix} -2 & 4 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & 17 \end{bmatrix} = U'.$$

Keeping track of multipliers gives $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$ and $A = LU'$ is an LU factorization. Factoring U' gives $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 17 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix}$.

6. (a) The equation $A\mathsf{x} = \mathsf{b}$ is $(LU)\mathsf{x} = \mathsf{b}$, that is, $L(U\mathsf{x}) = \mathsf{b}$. So let $\mathsf{y} = U\mathsf{x}$ and solve $L\mathsf{y} = \mathsf{b}$ for $\mathsf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then solve $U\mathsf{x} = \mathsf{y}$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. We solve $L\mathsf{y} = \mathsf{b}$ by forward substitution: $y_1 = 1$; $5y_1 + y_2 = 2$, so $y_2 = -3$; $-2y_1 + 3y_2 + y_3 = 3$, so $y_3 = 14$: we get $\mathsf{y} = \begin{bmatrix} 1 \\ -3 \\ 14 \end{bmatrix}$. Now we solve $U\mathsf{x} = \mathsf{c}$ by back substitution: $7x_3 = 14$, so $x_3 = 2$; $-5x_2 + x_3 = -3$, so $x_2 = 1$; $x_1 + 6x_2 + 7x_3 = 1$, so $x_1 = -19$: we get $\mathsf{x} = \begin{bmatrix} -19 \\ 1 \\ 2 \end{bmatrix}$. (Did you check your answer?)

   (b) Fact 2.1.33 is one of the most fundamental, useful and important properties in linear algebra. Since $A\mathsf{x} = \begin{bmatrix} 1 & 6 & 7 \\ 5 & 25 & 36 \\ -2 & -27 & -4 \end{bmatrix} \begin{bmatrix} -19 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathsf{b}$ (you did check your answer?), we have $\mathsf{b} = -19 \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ 25 \\ -27 \end{bmatrix} + 2 \begin{bmatrix} 7 \\ 36 \\ -4 \end{bmatrix}$.