1. (a) We are given that x is a linear combination of u and v. Thus there are scalars a and b with $\mathsf{x} = a\mathsf{u} + b\mathsf{v}$. A scalar multiple of x is a vector of the form $k\mathsf{x}$. This is $(ak)\mathsf{u} + (bk)\mathsf{v}$ which is also a linear combination of u and v, hence in the plane that they span.

   (b) Since x is in π, $ax_1 + bx_2 + cx_3 = 0$. A scalar multiple of x is a vector of the form $k\mathsf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}$. Since $a(kx_1) + b(kx_2) + c(kx_3) = k(ax_1 + bx_2 + cx_3) = 0$, $k\mathsf{x}$ is in π too.

2. (a) Let $P(1,1,1)$ be the given point. To find the distance from P to π, we find a point in the plane, say $Q(10,0,0)$, and project $\overrightarrow{PQ} = \begin{bmatrix} 9 \\ -1 \\ -1 \end{bmatrix}$ onto the normal $\mathsf{n} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. The desired distance is the length of this projection. We have $\operatorname{proj}_{\mathsf{n}} \overrightarrow{PQ} = \frac{8}{26}\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \frac{4}{13}\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. The desired distance is $\| \frac{4}{13}\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \| = \frac{4}{13}\sqrt{26}$.

   (b) Let $A(x,y,z)$ be the point of π closest to P. Then $\overrightarrow{PA}$ is the projection of PQ on n; that is, $\overrightarrow{PA} = \frac{4}{13}\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. We get $x = \frac{17}{13}$, $y = \frac{1}{13}$, $z = \frac{29}{13}$. The closest point is $A(\frac{17}{13}, \frac{1}{13}, \frac{29}{13})$.

3. (a) We find two nonparallel vectors u and v in the plane, find the projection p of u on v and take v and $\mathsf{u} - \mathsf{p}$ as our vectors. Let $\mathsf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\mathsf{p} = \operatorname{proj}_{\mathsf{v}} \mathsf{u} = \frac{\mathsf{u} \cdot \mathsf{v}}{\mathsf{v} \cdot \mathsf{v}} \mathsf{v} = \frac{-2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

   and $\mathsf{u} - \mathsf{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are suitable orthogonal vectors.
   [There are, of course, many correct answers to this question. Any two orthogonal vectors whose components satisfy the equation of the plane will do.]

   (b) In part (a), we learned that $\mathsf{e} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathsf{f} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal vectors spanning π, so we just write down the answer:

$$\operatorname{proj}_{\pi} \mathsf{w} = \frac{\mathsf{w} \cdot \mathsf{e}}{\mathsf{e} \cdot \mathsf{e}} \mathsf{e} + \frac{\mathsf{w} \cdot \mathsf{f}}{\mathsf{f} \cdot \mathsf{f}} \mathsf{f} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 4/3 \\ 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

© MMIV E. G. Goodaire, Mathematics and Statistics, Memorial University

4. (a) The lines have directions $\mathsf{d}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathsf{d}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Neither vector is a scalar multiple of the other, so the lines are not parallel. Suppose they intersect at (x, y, z). Then there would exist t and s such that

$$\begin{aligned} x &= -1 + 2t &&= 4 \\ y &= t &&= 1 + s \\ z &= 1 - 3t &&= -2 - s. \end{aligned}$$

Substituting $t = 1 + s$ in the first equation gives $s = \frac{3}{2}$, so $t = \frac{5}{2}$. Since these values do not satisfy the third equation, no such s and t exist, so the lines do not intersect.

(b) Each line is perpendicular to a normal to the plane, so the cross product of the two direction vectors is a normal:

$$\mathsf{n} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{j} \\ 2 & 1 & -3 \\ 0 & 1 & -1 \end{vmatrix} = 2\mathsf{i} + 2\mathsf{j} + 2\mathsf{k} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The plane has equation $x + y + z = d$, and since it contains the point $(-1, 0, 1)$, $d = 0$. The equation is $x + y + z = 0$.

5. The vector v is indeed a linear combination of $\mathsf{v}_1, \ldots, \mathsf{v}_5$, since $\mathsf{v} = 0\mathsf{v}_1 + 0\mathsf{v}_2 + (-1)\mathsf{v}_3 + 0\mathsf{v}_4 + 0\mathsf{v}_5$.

© MMIV E. G. Goodaire, Mathematics and Statistics, Memorial University