1. i. The system is $A\mathsf{x} = \mathsf{b}$ with $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathsf{b} = \begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix}$.

ii. We have $[A|I] = \left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right],$$

so $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$ and $\mathsf{x} = A^{-1}\mathsf{b} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \\ -2 \end{bmatrix}$.

iii. The given vectors are the columns of A, so $A\mathsf{x}$ is a linear combination of these vectors—recall 2.1.33. We have just shown that $A\begin{bmatrix} -5 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix}$, so

$$\begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix} = -5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

2. $A = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_1A \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_2E_1A$

$\to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3E_2E_1A \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_4E_3E_2E_1A = I$

where $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since $(E_4E_3E_2E_1)A = I$, $A = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$; which is

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. (a) $M = \begin{bmatrix} 19 & -10 & -6 \\ 14 & -11 & -2 \\ -2 & -5 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 19 & 10 & -6 \\ -14 & -11 & 2 \\ -2 & 5 & -3 \end{bmatrix}$,

$$AC^T = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 5 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} 19 & -14 & -2 \\ 10 & -11 & 5 \\ -6 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -23 & 0 & 0 \\ 0 & -23 & 0 \\ 0 & 0 & -23 \end{bmatrix}.$$

(b) $\det A = -23$

(c) $A^{-1} = -\dfrac{1}{23}\begin{bmatrix} 19 & -14 & -2 \\ 10 & -11 & 5 \\ -6 & 2 & -3 \end{bmatrix}$.

4. Since $C = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$, so $A^{-1} = \dfrac{1}{\det A}C^T = \dfrac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$.

Thus $A = (A^{-1})^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

5. Expanding by cofactors of the second column gives

$$\det A = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 7 - 1 - 5 = 1.$$

6. $\det A^TB^{-1}A^3(-B) = (\det A^T)(\det B^{-1})(\det A^3)(\det -B) = (\det A)\dfrac{1}{\det B}(\det A)^3(-\det B) = (2)\dfrac{-1}{5}(8)(5) = -16.$

7. (a) We compute $\det B$ using the Laplace expansion down the third column.

$$\det B = \begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = 10 + (-8) = 2.$$

Now we obtain (easily) that $\det \frac{1}{3}B = (\frac{1}{3})^3 \det B = \frac{2}{27}$ and $\det B^{-1} = \frac{1}{\det B} = \frac{1}{2}$.

(b) Let C be the cofactor matrix. We have $B = A^{-1} = \frac{1}{\det A}C^T$, so $C^T = (\det A)B$. Now $\det A = \frac{1}{\det A^{-1}} = \frac{1}{\det B} = \frac{1}{2}$. Thus $C^T = (\det A)B$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -2 & 0 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{3}{2} & -1 & 0 \\ -\frac{1}{2} & 2 & \frac{1}{2} \end{bmatrix} \text{ and } C = (C^T)^T = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 1 & -1 & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

8. $$\begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 0 & -10 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = 5\begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= -5\begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = -5\begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix}$$

$$= -5\begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix} = -5\begin{vmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 & \frac{7}{4} \end{vmatrix} = -5(-4)(\tfrac{7}{4}) = 35.$$

9. $$\begin{vmatrix} 2p & -a+u & 3u \\ 2q & -b+v & 3v \\ 2r & -c+w & 3w \end{vmatrix} = \begin{vmatrix} 2p & 2q & 2r \\ -a+u & -b+v & -c+w \\ 3u & 3v & 3w \end{vmatrix}$$

$$= 2(3)\begin{vmatrix} p & q & r \\ -a+u & -b+v & -c+w \\ u & v & w \end{vmatrix} = 6\begin{vmatrix} p & q & r \\ -a & -b & -c \\ u & v & w \end{vmatrix} = -6\begin{vmatrix} p & q & r \\ a & b & c \\ u & v & w \end{vmatrix}$$

$$= -(-6)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 6(5) = 30.$$

10. v_1 is not an eigenvector: an eigenvector is, by definition, **not zero**.

$A\mathsf{v}_2 = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$ is not $\lambda\mathsf{v}_2$ for any scalar λ, so v_2 is not an eigenvector.

$A\mathsf{v}_3 = \mathsf{v}_3$, so v_3 is an eigenvector of A.

$A\mathsf{v}_4 = -2\mathsf{v}_4$, so v_4 is an eigenvector of A.

$A\mathsf{v}_5 = 5\mathsf{v}_5$, so v_5 is an eigenvector of A.

$A\mathsf{v}_6 = \begin{bmatrix} 12 \\ 8 \\ 6 \end{bmatrix}$ is not $\lambda\mathsf{v}_6$ for any scalar λ, so v_6 is not an eigenvector.

11. (a) The characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

The eigenvalues of A are -1 and 4. To find the eigenspace for $\lambda = -1$, we solve the homogeneous system $(A-\lambda I)\mathsf{x} = \mathsf{0}$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. With $\lambda = -1$, we have

$$A - \lambda I = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The solutions are $x_2 = t$, $x_1 = -t$. The corresponding eigenspace consists of vectors of the form $\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find the eigenspace for $\lambda = 4$, we solve the homogeneous system $(A-\lambda I)\mathsf{x} = \mathsf{0}$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. With $\lambda = 4$, we have

$$A - \lambda I = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \to \begin{bmatrix} -1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

The solutions are $x_2 = t$, $x_1 = \frac{2}{3}t$. The corresponding eigenspace consists of vectors of the form $\begin{bmatrix} \frac{2}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = s\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(b) The characteristic polynomial of $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$ is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 & 3 \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda - 8 = -(\lambda - 2)^3.$$

The only eigenvalue of A is $\lambda = 2$. To find the corresponding eigenspace, we solve the homogeneous system $(A - \lambda I)\mathsf{x} = 0$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $\lambda = 2$. We have

$$A - \lambda I = \begin{bmatrix} -1 & -2 & 3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The solutions are $x_3 = t$, $x_2 = s$, $x_1 = -2s + 3t$. The eigenspace consists of vectors of the form $\begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

12. We are given $A\mathsf{x} = \lambda\mathsf{x}$ with $\mathsf{x} \neq 0$. Since A is invertible, $A\mathsf{x} \neq 0$, so $\lambda \neq 0$. Multiplying $A\mathsf{x} = \lambda\mathsf{x}$ by A^{-1} gives $\mathsf{x} = \lambda A^{-1}\mathsf{x}$, so $A^{-1}\mathsf{x} = \frac{1}{\lambda}\mathsf{x}$. Thus x is also an eigenvector of A^{-1}, and with eigenvalue $\frac{1}{\lambda}$.