Am I ready for Final? Mathematics 2050 Winter 2010

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- **Final Exam will be on April 16 at 12-2 pm in PE 2000**

- **Please, have a picture ID with you.**

- **Use of calculators capable of symbolic manipulations is not allowed during the exam.**

- **Formulas marked with $(*)$ were not covered this year so you are not expected to know them for Final Exam.**

I Geometry of plane and 3D space.

We will use the following notations:

points $P_0(x_0, y_0, z_0)$, $P_1(x_1, y_1, z_1)$; origin $O(0,0,0)$; arbitrary point $P(x,y,z)$. Then arbitrary vector $\vec{x} = \vec{OP}$.

Let vector $\vec{u} = \vec{OP_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\vec{d} = \vec{P_0P_1} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix}$.

Line through point $P_0(x_0, y_0, z_0)$ with the direction vector $\vec{d}$ has equation

$$\vec{x} = \vec{u} + t\vec{d}$$

This is the line passing through points P_1 and P_0.

Planes orthogonal to $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ have equations

$$n_1x + n_2y + n_3z = k, \quad k = \text{constant}$$

For example, $k = 0$ if the plane contains the origin.

A point P_2 belongs to a plane or a line if and only if its coordinates obey corresponding equation.

Operations with vectors:

0) if $a \geq 0$ is a number then $a\vec{v}$ is a vector in the same direction as $\vec{v}$; if $a \leq 0$ is a number then $a\vec{v}$ is a vector in the opposite direction to vector $\vec{v}$;

1) linear combination of two vectors $a\vec{v} + b\vec{u}$ is a vector; the set of all linear combinations of two vectors is called *span* of these vectors.

2) dot product $\vec{v} \cdot \vec{u}$ is a number $v_1u_1 + v_2u_2 + v_3u_3$;

Important properties: $\vec{v} \cdot \vec{u} = 0$ if and only if the vectors are orthogonal.

$\cos(\widehat{\vec{u}, \vec{v}}) = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||}$

3) cross product $\vec{w} = \vec{u} \times \vec{v}$ is a vector with components

$$w_1 = v_2u_3 - v_3u_2, \quad w_2 = -(v_1u_3 - v_3u_1), \quad w_3 = v_1u_2 - v_2u_1.$$

Important properties: $\vec{w} \cdot \vec{u} = \vec{w} \cdot \vec{v} = 0$ and $||\vec{w}||$ is the area of the parallelogram with sides $\vec{v}, \vec{u}$.

4) projection of $\vec{u}$ onto $\vec{v}$ is a vector $a\vec{v}$, where number a is found by the formula: $a = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$.

Important inequalities:
Cauchy-Bunyakovsky-Schwartz inequality: $|\vec{u}\cdot\vec{v}| \le ||\vec{u}||\,||\vec{v}||$
Triangular inequality $||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$

II Linear system of equations.

Consider non-homogeneous system $AX = B$, where A is a $n \times m$-matrix of coefficients, B is $n \times 1$ non-zero matrix of constants, and X is $m \times 1$-matrix of unknowns.

If $m = n$ and $\det A \neq 0$ then the system has a unique vector solution X. (in this and only this case inverse matrix A^{-1} does exist and $X = A^{-1}B$.)

If $\det A = 0$ the system may have no solutions or infinitely many solutions (parametric solution).

A homogeneous system $AX = \mathbf{0}$ has non-zero solutions if and only if $\det A = 0$. Otherwise it has a unique solution $X = \mathbf{0}$. (Here $\mathbf{0}$ denotes a column of zeroes).

How to find the solution of $AX = B$ if it exists?

Way 1 Use Elementary Row Operations (also known as Gaussian elimination) to rewrite the system in the Row Echelon Form. If the form is "perfect stair case" then the system has a unique solution. Use back substitution to find it.

If the Row Echelon Form is not perfect stair case and gives you a contradiction (e.g.equation $0 = 1$) then the system does not have a solution. Otherwise the system has infinitely many solutions. Total number of variables minus rank of A gives you the number of parameters in the parametric form of the solution. (Rank of A is the number of leading 1's in the REF of the matrix A.)

Way 2 If A^{-1} is known than $X = A^{-1}B$ (unique solution).

$(*)$ **Way 3** Cramer's Rule. (applicable only if $\det A \neq 0$).

Way 4 If LU-factorization of the matric A is given then solve two triangular systems to find the solution of $LU\vec{x} = \vec{b}$. First solve $L\vec{y} = \vec{b}$. Then $U\vec{x} = \vec{y}$. Here you take advantage from the fact that it is easy to solve a triangular system compare to general one.

How to find $\det A$*?*

Way 1 Cofactor expansion.

$$\det A = \sum_{k=1}^{n} a_{ik}c_{ik},$$

where a_{ik} are elements of the row i of A, and $c_{ik} = (-1)^{i+k}m_{ik}$ are corresponding cofactors. Note that you can use any row, so try to pick the one which has more zero entries.

Way 2 Rewrite A as a upper triangular matrix U using only 3rd elementary row operation $R_i \to R_i + kR_j$. Then $\det A$ is product of diagonal entries of U. If you use other elementary row operation, remember how they effect the value of $\det A$, and adjust accordingly.

Also, remember properties of det :

$\det kA = k^n \det A$, $\det A^T = \det A$, $\det A^{-1} = (\det A)^{-1}$, $\det(AB) = \det A \det B$.

How to find A^{-1}?

Way 1 Set $[A|I]$ and do elementary row operations (also known as Gaussian elimination) to get $[I|B]$. Then $B = A^{-1}$.

Way 2 Find matrix of cofactors C for A. Note that $AC^T = kI$, where $k = \det A$. Then $A^{-1} = (1/k)C^T$.

Way 3 For some matrices such as *diagonal* or *elementary* matrices it is very easy to find an inverse (remember how?).

($*$) One may take a decomposition of A in terms of elementary matrices and then use the property $(EF)^{-1} = F^{-1}E^{-1}$.

How to find LU factorization.

Rewrite A as an upper triangular matrix U using only 3rd elementary row operation. Then L would have all 1's on its diagonal and below diagonal entries L_{ij} are the numbers used for the row reduction.

($*$)Remember that for symmetric matrices ($A = A^T$) $L = U^T$.

II Eigenvalues and eigenspaces.

Let $AX = \lambda X$ and $X \neq \mathbf{0}$. Then number λ is eigenvalue of A and X is an eigenvector.

Any $n \times n$ matrix has exactly n (not necessarily distinct) eigenvalues.

They are roots of algebraic equation $\det(A - \lambda I) = 0$. For each λ the corresponding eigenspace is a solution of the homogeneous system $(A - \lambda I)\vec{x} = \vec{0}$.

If all eigenvalues of A are distinct we have $P^{-1}AP = D$, where D is a diagonal matrix whose entries are the eigenvalues of A, and P is an invertible matrix whose columns are (linearly independent) eigenvectors of A.

In such situation we say that A is *similar* to D and is *diagonalizable*.

Not all matrices are diagonalizable. The one which have eigenvalues of multiplicity 2 or greater may have less that n linearly independent eigenvectors and this would prevent existence of P^{-1}.

Applications of diagonalization.

If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$. This is an efficient way to find matrix in a (big) power $k > 1$, because D^k is easy to find.

Let A define a dynamical system $\vec{V}_{k+1} = A\vec{V}_k$ with given initial vector $\vec{V}_0$. Than $\vec{V}_k = A^k\vec{V}_0$.

(*) Using the above way to get A^k we find $\vec{V}_k = \sum_{j=1}^{n} \lambda_j^k \vec{X}_j b_j$, where λ_j, $j = 1, 2, 3, ..., n$ are eigenvalues with corresponding eigenvectors $\vec{X}_j$, and b_j are constant found by $\vec{b} = P^{-1}\vec{V}_0$.

A matrix A is called *stochastic* if $0 \leq A_{ij} \leq 1$ and the columns of A all some to one.

A stochastic matric is called *regular* if some power A^m of A has all positive elements.

A stochastic matrix always has eigenvalue 1.

If a stochastic matrix is also regular, then it has eigenvector S with all positive component summing to one: $S_j > 0$, $\sum_{j=1}^{n} S_j = 1$. This vector is called a *steady vector* of the Markov's process $\vec{V}_{k+1} = A\vec{V}_k$, defined by the stochastic matrix A.

The steady vector is the limit of the sequence of the systems' state vectors: $\lim_{k\to\infty} \vec{V}_k = S$. If the stochastic matrix gives transition probabilities between n possible states of a system then the steady vector S shows the proportion of time the system spends in each of the these states over a long period of time.

Calculations and questions:

0) find length of given vector $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

1) given two vectors, find their sum using the triangular or parallelogram rule.

2) find angle between two vectors (use: dot product). Remember special angles: $\cos 0^o = 1, \cos 30^0 = \sqrt{3}/2, \cos 45^o = \sqrt{2}/2, \cos 60^o = 1/2, \cos 90^o = 0$.

3) find area of triangle with sides $\vec{v}$ and $\vec{u}$ (may use cross product)

4) find equation of the plane containing three given points (use cross product to find the normal vector $\vec{n} = \vec{AB} \times \vec{AC}$)

5) find equation of the plane containing two given intersecting lines (use: cross product $\vec{n} = \vec{d_1} \times \vec{d_2}$)

6) find equation of plane containing given line and given point

7) given two vectors, find two orthogonal vectors in the same plane (hint: dot product of the orthogonal vectors must be zero)

8) find projection of a given vector onto a plane

9) find the distance from given point to another given point

10) find the distance from given point to given line (use: projection and pythagorean theorem)

11) find the point in the plane closest to the given point (hint: find the intersection of the plane and the line orthogonal to this plane through the point)

12) find the distance from given point to given plane (use: either projection or the point from question 11)

13) find the distance from a plane to a line which is parallel to it.

($*$) find equation of the line of intersection of two planes (use: vector product)

14) find the point of intersection of two lines given by their parametric equations.

15) determine if given vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent. (solve homogeneous system of equations $\vec{0} = x\vec{u} + y\vec{v} + z\vec{w}$; if it has a non-zero solution then the vectors are linearly dependent).

16) express given vector $\vec{b}$ as a linear combination of given vectors $\vec{u}, \vec{v}, \vec{w}$. (solve linear system of equation $\vec{b} = x\vec{u} + y\vec{v} + z\vec{w}$).

17) determine if a given vector $\vec{b}$ is in the span of $\vec{u}, \vec{v}, \vec{w}$. (this is similar to question 16: if linear system of equation $\vec{b} = x\vec{u} + y\vec{v} + z\vec{w}$ has no solutions then the answer is NO.)

18) Solve a system of equation AX=B by converting it in the Row Echelon Form; (in case of parametric solution, write the general solution in the vector form, for example: $\vec{X} = \vec{X}_0 + t\vec{v} + s\vec{u}$, $t, s \in R$.

19) Given a system of equation with algebraic coefficients, (e.g. $((a^2 - 1)z = (a + b))$ find for which values of this coefficients (a and b) the system has no solution, unique solution and infinitely many solutions.

20) Find the inverse matrix and its determinant.

21) Given the value of determinant for some matrix, find the determinant of a modifies matrix.

22) Write given matrix as a product of elementary matrices. (hint: convert the matrix to the identity matrix and keep track of the elementary row operations $E_3E_2E_1A = I$ then solve for $A = E_1^{-1}E_2^{-1}E_3^{-1}$.

23) Find LU factorization of a matrix and use it to solve a system of linear equations.

24) Find a diagonal matrix similar to the given matrix (hint: find eigenvalues and eigenvectors of A).

25) Find a steady vector for a Markov's process and the time the system spends in each state in the long run.

26) Do simple proofs involving symbolic manipulations with matrices and knowing the rules such as $(AB)^T = B^TA^T$, $(AB)^{-1} = B^{-1}A^{-1}$ etc.

Please, make sure that you understand and **can do** all the problems from midterm tests as well as from your homework assignments.
Send me an e-mail if you need something extra or if you have a question:
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Good luck!