MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Final Examination Solutions	**Mathematics 2050** **Drs. Bahturin, Goodaire and Zhao**	Fall 2003

Time: 2 hours.

All answers should be justified using good English.

1. Let $\mathsf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$.

[2] (a) Find $5\mathsf{u} - 3\mathsf{v}$.

[2] (b) Find $\mathsf{u} \cdot \mathsf{v}$.

[3] (c) Find $\|\mathsf{u} + \mathsf{v}\|$.

[3] (d) Find the exact value of the cosine of the angle between u and v.
[Most calculators do not give exact answers.]

Solution.

(a) $5\mathsf{u} - 3\mathsf{v} = 5\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 0 \end{bmatrix}$.

(b) $\mathsf{u} \cdot \mathsf{v} = 1(-3) + 2(0) + 3(5) = 12$.

(c) $\mathsf{u} + \mathsf{v} = \begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix}$, so $\|\mathsf{u} + \mathsf{v}\| = \sqrt{(-2)^2 + 2^2 + 8^2} = 6\sqrt{2}$.

(d) $\cos\theta = \dfrac{\mathsf{u} \cdot \mathsf{v}}{\|\mathsf{u}\|\,\|\mathsf{v}\|} = \dfrac{12}{\sqrt{14}\sqrt{34}} = \dfrac{6}{\sqrt{119}}$.

[3] 2. (a) Find the equation of the line ℓ through $A(1,2,3)$ and parallel to $\mathsf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

[6] (b) Find all points B on ℓ such that $\overrightarrow{AB}$ is a unit vector.

Solution.

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

(b) Let $B = (x, y, z)$ Since B is on ℓ, $\overrightarrow{AB} = \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 2t \end{bmatrix}$. We wish $\|\overrightarrow{AB}\| = 1$. Thus $\sqrt{(-t)^2 + t^2 + (2t)^2} = 1$, so $6t^2 = 1$, $t^2 = \frac{1}{6}$ and $t = \pm\frac{1}{\sqrt{6}}$. There are two points: $B_1 = (1 - \frac{\sqrt{6}}{6}, 2 + \frac{\sqrt{6}}{6}, 3 + \frac{2\sqrt{6}}{6})$ and $B_2 = (1 + \frac{\sqrt{6}}{6}, 2 - \frac{\sqrt{6}}{6}, 3 - \frac{2\sqrt{6}}{6})$.

[6] 3. (a) Find the equation of the plane π through $A(-2, -1, 1)$, $B(-1, 1, 2)$, $C(0, 1, 3)$.

[4] (b) Find the distance from $D(2, 2, -1)$ to π.

Solution.

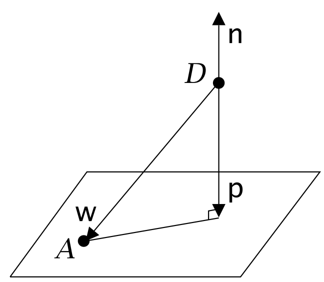
(a) A normal n to π is $\overrightarrow{AB} \times \overrightarrow{AC}$. Since $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$$\mathsf{n} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2(\mathsf{i} - \mathsf{k}) = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is also a normal, so the plane has equation of the form $x - z = d$. Since A is on the plane, $x - z = -3$.

(b)

The required distance is the length of the projection p of $\mathsf{w} = \overrightarrow{DA}$ on the normal $\mathsf{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Since, $\mathsf{w} = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$, $\text{proj}_{\mathsf{n}}\,\mathsf{w} = \dfrac{\mathsf{w} \cdot \mathsf{n}}{\mathsf{n} \cdot \mathsf{n}}\,\mathsf{n} = -3\mathsf{n}$. The required distance is $3\|\mathsf{n}\| = 3\sqrt{2}$.



[5] 4. Suppose u, v, and w are linearly dependent vectors in R^n. Prove that $\sqrt{2}\mathsf{u}$, $-\mathsf{v}$, and $\frac{1}{3}\mathsf{w}$ are also linearly dependent.

Solution. Since u, v, w are linearly dependent, there are scalars c_1, c_2, c_3, not all 0, such that $c_1\mathsf{u} + c_2\mathsf{v} + c_3\mathsf{w} = \mathsf{0}$. Thus,

$$\frac{c_1}{\sqrt{2}}(\sqrt{2}\mathsf{u}) + (-c_2)(-\mathsf{v}) + (3c_3)(\frac{1}{3}\mathsf{w}) = \mathsf{0}.$$

Since not all the coefficients here, $\frac{c_1}{\sqrt{2}}$, $-c_2$, $3c_3$ are 0, the vectors $\sqrt{2}\mathsf{u}$, $-\mathsf{v}$, $\frac{1}{3}\mathsf{w}$ are linearly dependent.

[6] 5. (a) Solve the system

$$\begin{aligned} x_2 + x_3 - x_4 &= 1 \\ 2x_1 + 3x_3 + x_4 &= 2 \end{aligned} \qquad (*)$$

expressing the solution as a vector which is the sum of a particular solution x_p to (*) and a solution x_h of the corresponding homogeneous system. Identify x_p and x_h.

[3] (b) Write $\mathsf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of the columns of $A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 2 & 0 & 3 & 1 \end{bmatrix}$ **using specific coefficients.**

Solution.

(a) The system is $A\mathsf{x} = \mathsf{b}$ with $A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 2 & 0 & 3 & 1 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathsf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Gaussian elimination on the augmented matrix is very easy:

$$[A|\mathsf{b}] = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 1 \\ 2 & 0 & 3 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right].$$

We have $x_3 = t$ and $x_4 = s$ free. Back substitution gives $x_2 = 1 - x_3 + x_4 = 1 - t + s$ and $2x_1 = 2 - 3x_3 - x_4 = 2 - 3t - s$, so $x_1 = 1 - \frac{3}{2}t - \frac{1}{2}s$. The solution is

$$\mathsf{x} = \begin{bmatrix} 1 - \frac{3}{2}t - \frac{1}{2}s \\ 2 - 3t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

which is $\mathsf{x} = \mathsf{x}_p + \mathsf{x}_h$ with $\mathsf{x}_p = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ a particular solution and $\mathsf{x}_h = t\begin{bmatrix} -\frac{3}{2} \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix}$ a solution of the corresponding homogeneous system.

(b) Part (a) tells us that $A\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

6. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, find each of the following, if possible. If not possible, explain why not.

[2] (a) $A + 2C$

[2] (b) $(AB)^T$

[2] (c) $B^T A^T$

[2] (d) $AC + B$

Solution.

(a) $A + 2C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 2 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 8 & 7 & 6 \end{bmatrix}$.

(b) Since $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ 7 & 26 \end{bmatrix}$, $(AB)^T = \begin{bmatrix} 4 & 7 \\ 11 & 26 \end{bmatrix}$.

(c) $B^T A^T = (AB)^T = \begin{bmatrix} 4 & 7 \\ 11 & 26 \end{bmatrix}$.

(d) $AC+B$ is not defined since the matrix product AC is not defined; A is 2×3, so C should be $3 \times n$, but C is 2×3.

[6] 7. Find an LU decomposition of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R3 \to R3 - 7(R1)]{R2 \to R2 - 4(R1)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{R3 \to R3 - 2(R2)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$$

with $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$, the matrix that records the multipliers.

[6] 8. Write $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as the product of elementary matrices.

Solution.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = E_1 A \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = E_2(E_1 A) \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_3(E_2 E_1 A)$$

with $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Since $(E_3 E_2 E_1)A = I$,

$$A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

[A different sequence of row operations leads to $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.]

[6] 9. Given 4×4 matrices X, Y, Z, with Y invertible, express $\det(-2X^T Y^{-1} Z^2)$ in terms of $\det X$, $\det Y$, and $\det Z$.

Solution. $\det(-2X^T Y^{-1} Z^2) = (-2)^4 \det X \dfrac{1}{\det Y} (\det Z)^2 = \dfrac{16 \det X (\det Z)^2}{\det Y}$.

[6] 10. (a) Find the inverse of $A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -5 & -2 \\ 3 & 14 & 4 \end{bmatrix}$.

[4] (b) Use your answer to (a) to solve the system
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -2x_1 - 5x_2 - 2x_3 &= 1 \\ 3x_1 + 14x_2 + 4x_3 &= 1. \end{aligned}$$

Solution.

(a) $$[A|I] = \left[\begin{array}{rrr|rrr} 1 & 3 & 1 & 1 & 0 & 0 \\ -2 & -5 & -2 & 0 & 1 & 0 \\ 3 & 14 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & -5 & -3 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -13 & -5 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 8 & 2 & -1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -13 & -5 & 1 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 8 & 2 & -1 \\ 2 & 1 & 0 \\ -13 & -5 & 1 \end{bmatrix}$.

(b) The system is $A\mathsf{x} = \mathsf{b}$ with $\mathsf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so the solution is

$$\mathsf{x} = A^{-1}\mathsf{b} = \begin{bmatrix} 8 & 2 & -1 \\ 2 & 1 & 0 \\ -13 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -17 \end{bmatrix}.$$

[6] 11. Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ -1 & -4 & 3 & 1 \\ 1 & 6 & 4 & 2 \end{bmatrix}$ by reducing to a triangular matrix.

Solution.

$$\begin{vmatrix} 0 & 2 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ -1 & -4 & 3 & 1 \\ 1 & 6 & 4 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 1 & 1 \\ -1 & -4 & 3 & 1 \\ 1 & 6 & 4 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 4 & 5 & 4 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -12.$$

[3] 12. (a) Define the term *similar matrices*.

Now let $A = \begin{bmatrix} -2 & -4 \\ 3 & 6 \end{bmatrix}$.

[5] (b) Find the characteristic polynomial of A and use this to explain why A is similar to a diagonal matrix.

[5] (c) Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

[2] (d) Find a diagonal matrix D and an invertible matrix S such that $S^{-1}DS = A$.

Solution.

(a) Matrices A and B are *similar* if there exists an invertible matrix P such that $B = P^{-1}AP$.

(b) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -4 \\ 3 & 6-\lambda \end{vmatrix} = -(2+\lambda)(6-\lambda) + 12 = \lambda^2 - 4\lambda = \lambda(\lambda - 4).$$

Since the eigenvalues, $\lambda = 0, 4$ are distinct, A is similar to a diagonal matrix.

(c) To find the eigenspace for $\lambda = 0$, we solve the system $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = 0$ and $\mathsf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. We have

$$A \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

so $y = t$ is free, $x = -2y = -2t$ and $\mathsf{x} = t\begin{bmatrix}-2\\1\end{bmatrix}$. To find the eigenspace for $\lambda = 4$, we solve the system $(A - \lambda I)\mathsf{x} = \mathsf{0}$ with $\lambda = 4$ and $\mathsf{x} = \begin{bmatrix}x\\y\end{bmatrix}$. We have

$$A - 4I = \begin{bmatrix} -6 & -4 \\ 3 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$$

so $y = t$ is free, $3x = -2y = -2t$, $x = -\frac{2}{3}t$ and $\mathsf{x} = t\begin{bmatrix}-\frac{2}{3}\\1\end{bmatrix}$. The vectors $\begin{bmatrix}-2\\1\end{bmatrix}$ and $\begin{bmatrix}-2\\3\end{bmatrix}$ are eigenvectors for 0 and 4, respectively. Putting these into the columns of a matrix $P = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}$, we have $P^{-1}AP = D = \begin{bmatrix}0 & 0\\0 & 4\end{bmatrix}$, the diagonal matrix whose diagonal entries correspond, in order, to the eigenvectors which are the columns of P.

(d) Since $P^{-1}AP = D$, $A = PDP^{-1}$, so take $S = P^{-1} = \frac{1}{4}\begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$.

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[100]