# Extra Problems for Math 2050

## Linear Algebra I

1. Find the vector $\overrightarrow{AB}$ and illustrate with a picture if $A = (-2, 1)$ and $B = (1, 4)$.

2. Find $B$, given $A = (1, 4)$ and $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. $A = (1, 4)$ and $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

3. If possible, express $\mathsf{x} = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}$ as a scalar multiple of $\mathsf{u} = \begin{bmatrix} 8 \\ -28 \\ -16 \end{bmatrix}$, as a scalar multiple of $\mathsf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and as a scalar multiple of $\mathsf{w} = \begin{bmatrix} \frac{2}{7} \\ -1 \\ -\frac{4}{7} \end{bmatrix}$. Justify your answers.

4. Given $\mathsf{u}_1 = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}$, $\mathsf{u}_2 = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix}$, $\mathsf{u}_3 = \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix}$, $\mathsf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathsf{u}_5 = \begin{bmatrix} -5 \\ 10 \\ -20 \end{bmatrix}$, list all pairs of parallel vectors.

5. Find the indicated vectors.

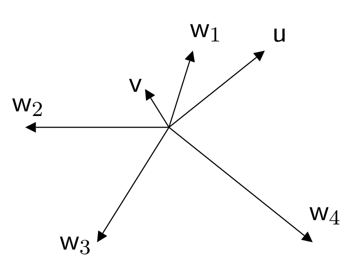
   (a) $4\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (b) $3\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

6. Suppose $\mathsf{u} = \begin{bmatrix} a \\ -5 \end{bmatrix}$, $\mathsf{v} = \begin{bmatrix} 1 \\ 6-b \end{bmatrix}$ and $\mathsf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $a$ and $b$ if $2\mathsf{u} - \mathsf{v} = \mathsf{w}$.

7. Suppose $\mathsf{x}$, $\mathsf{y}$, $\mathsf{u}$, and $\mathsf{v}$ are vectors. Express $\mathsf{u}$ and $\mathsf{v}$ in terms of $\mathsf{x}$ and $\mathsf{y}$ given $\mathsf{x} - \mathsf{y} = \mathsf{u}$ and $2\mathsf{x} + 3\mathsf{y} = \mathsf{v}$.

8. Given $\mathsf{u} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, find $\mathsf{u} - \mathsf{v}$ and illustrate with a picture like that in Figure 6.

9. Shown at the right are two nonparallel vectors $\mathsf{u}$ and $\mathsf{v}$ and four other vectors $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3, \mathsf{w}_4$. Reproduce $\mathsf{u}$, $\mathsf{v}$, and $\mathsf{w}_1$ in a picture by themselves and exhibit $\mathsf{w}_1$ as the diagonal of a parallelogram with sides parallel to $\mathsf{u}$ and $\mathsf{v}$. Guess (approximate) values of $a$ and $b$ so that $\mathsf{w}_1 = a\mathsf{u} + b\mathsf{v}$.




10. Express $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

11. Is it possible to express $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$? Explain.

12. Let $\mathsf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathsf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathsf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

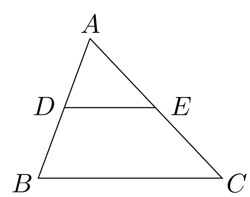
    Is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ a linear combination of $\mathsf{v}_1$, $\mathsf{v}_2$, $\mathsf{v}_3$?

13. Is it possible to express the zero vector as a linear combination of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -8 \end{bmatrix}$ in more than one way? Justify your answer.

14. Suppose $\mathsf{u}$ and $\mathsf{v}$ are vectors. Show that any linear combination of $2\mathsf{u}$ and $-3\mathsf{v}$ is a linear combination of $\mathsf{u}$ and $\mathsf{v}$.

15. Let $O(0,0)$, $A(2,2)$, $B(4,4)$, $C(2,-1)$, $D(4,1)$, $E(6,3)$, $F(2,-4)$, $G(6,0)$, $H(8,2)$ be the indicated points in the $xy$-plane. Let $\mathsf{u} = \overrightarrow{OA}$ be the vector pictured by the arrow from $O$ to $A$ and $\mathsf{v} = \overrightarrow{OC}$ the vector pictured by the arrow from $O$ to $C$. Find $\mathsf{u}$ and $\mathsf{v}$. Then express the vector $\overrightarrow{OD}$ from 0 to $D$ as a combination of $\mathsf{u}$ and $\mathsf{v}$. For example, $\overrightarrow{OE} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 2\mathsf{u} + \mathsf{v}$.

16. Suppose $ABC$ is a triangle. Let $D$ be the midpoint of $AB$ and $E$ be the midpoint of $AC$. Use vectors to show that $DE$ is parallel to $BC$ and half its length.




17. Let $A = (1,2)$, $B = (4,-3)$ and $C = (-1,-2)$ be three points in the Euclidean plane. Find a fourth point $D$ such that the $A$, $B$, $C$ and $D$ are the vertices of a parallelogram and justify your answer. Is the answer unique?

18. Prove that vector addition is closed and commutative and every vector $\mathsf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ has a negative $-\mathsf{u}$ which has the property that $\mathsf{u} + (-\mathsf{u}) = (-\mathsf{u}) + \mathsf{u} = \mathsf{0}$.

19. Show that any linear combination of $\begin{bmatrix} 1 \\ \frac{3}{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ is also a linear combination of $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

20. Explain how one could use vectors to show that three points $X$, $Y$, and $Z$ are *collinear*. (Points are collinear if they lie on a line.)

21. If $\mathsf{u}$ and $\mathsf{v}$ are parallel, show that some *nontrivial* linear combination of $\mathsf{u}$ and $\mathsf{v}$ is the zero vector. (*Nontrivial* means that $0\mathsf{u} + 0\mathsf{v} = \mathsf{0}$ doesn't count!)

22. Let $\mathsf{u} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Find a unit vector in the direction of $\mathsf{u}$ and a vector of length 2 in the direction opposite to $\mathsf{v}$.

23. Susie is asked to find the length of the vector $\mathsf{u} = \frac{4}{7}\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. She proceeds like this.

$$\mathsf{u} = \begin{bmatrix} -\frac{8}{7} \\ \frac{4}{7} \\ \frac{20}{7} \end{bmatrix}, \text{ so } \|\mathsf{u}\| = \sqrt{(\tfrac{8}{7})^2 + (\sqrt{\tfrac{4}{7}})^2 + (\tfrac{20}{7})^2} = \sqrt{4807}.$$

Do you like this approach?

24. Find the angle between each of the following pairs of vectors. Give your answers in radians and in degrees. If it is necessary to approximate, give radians to two decimal places of accuracy and degrees to the nearest degree.

(a) $\mathsf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathsf{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ (b) $\mathsf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathsf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

25. Find $\mathsf{u} \cdot \mathsf{v}$ given $\|\mathsf{u}\| = 1$, $\|\mathsf{v}\| = 3$, and the angle between $\mathsf{u}$ and $\mathsf{v}$ is $45°$.

26. Can $\mathsf{u} \cdot \mathsf{v} = -7$ if $\|\mathsf{u}\| = 3$ and $\|\mathsf{v}\| = 2$? Justify your answer.

27. Let $\mathsf{u} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\mathsf{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and $\mathsf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find $\|\mathsf{u}\|$, $\|\mathsf{u} + \mathsf{v}\|$ and $\left\| \frac{\mathsf{w}}{\|\mathsf{w}\|} \right\|$.

(b) Find a unit vector in the same direction as $\mathsf{u}$.

(c) Find a vector of norm 4 in the direction opposite to $\mathsf{v}$.

28. Let $\mathsf{u}$ and $\mathsf{v}$ be vectors of lengths 3 and 5 respectively and suppose that $\mathsf{u} \cdot \mathsf{v} = 8$. Find $(\mathsf{u} - \mathsf{v}) \cdot (2\mathsf{u} - 3\mathsf{v})$ and $\|\mathsf{u} + \mathsf{v}\|^2$.

29. Let $\mathsf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find all numbers $k$ such that $\mathsf{u} + k\mathsf{v}$ has norm 3.

30. If a nonzero vector $\mathsf{u}$ is orthogonal to another vector $\mathsf{v}$ and $k$ is a scalar, show that $\mathsf{u} + k\mathsf{v}$ is not orthogonal to $\mathsf{u}$.

31. Given the points $A(-1, 0)$ and $B(2, 3)$ in the plane, find all points $C$ such that $A$, $B$, and $C$ are the vertices of a right angle triangle with right angle at $A$, and $AC$ of length 2.

32. Find a point $D$ such that $(1, 2)$, $(-3, -1)$, $(4, -2)$ and $D$ are the vertices of a square and justify your answer.

33. Let $\mathsf{u} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, $\mathsf{w} = \begin{bmatrix} -2 \\ 5 \\ 7 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$. Show that $\mathsf{u}$ is orthogonal to $\mathsf{v}$ and to $\mathsf{w}$.

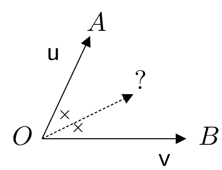
34. Assuming all vectors 2-dimensional, prove that the dot product is commutative and that $\mathsf{v} \cdot \mathsf{v} = 0$ if and only if $\mathsf{v} = 0$.

35. Verify the Cauchy–Schwarz inequality for $\mathsf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

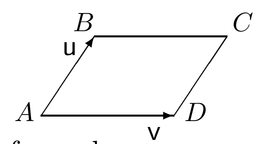
36. Use the Cauchy–Schwarz inequality to prove that $(a\cos\theta + b\sin\theta)^2 \le a^2 + b^2$ for all $a, b \in \mathsf{R}$.

37. Let $a, b, c, d$ be real numbers. Use the Cauchy–Schwarz inequality to prove that $(3ac + bd)^2 \le (3a^2 + b^2)(3c^2 + d^2)$.

38. Suppose vectors $\mathsf{u}$ and $\mathsf{v}$ are represented by arrows $OA$ and $OB$ in the plane, as shown. Find a formula for a vector whose arrow bisects angle $AOB$.




39. Let $\mathcal{P}$ be a parallelogram with sides representing the vectors $\mathsf{u}$ and $\mathsf{v}$ as shown.




Express the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$ of $\mathcal{P}$ in terms of $\mathsf{u}$ and $\mathsf{v}$.

40. Express the plane with equation $3x - 2y + z = 0$ as the set of all linear combinations of two nonparallel vectors. (There are many possibilities!)

41. Find the equation of each of the plane parallel to the plane with equation $4x+2y-z = 7$ and passing through the point $(1, 2, 3)$.

42. Verify that $\mathsf{u} \times \mathsf{v} = -(\mathsf{v} \times \mathsf{u})$ given $\mathsf{u} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$.

43. Given $\mathsf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, find the cross product $\mathsf{u} \times \mathsf{v}$, then verify that this vector is perpendicular to $\mathsf{u}$ and to $\mathsf{v}$ and that $\mathsf{v} \times \mathsf{u} = -(\mathsf{u} \times \mathsf{v})$.

44. Given $\mathsf{u} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathsf{v} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, and $\mathsf{w} = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$, compute $(\mathsf{u} \times \mathsf{v}) \times \mathsf{w}$ and $\mathsf{u} \times (\mathsf{v} \times \mathsf{w})$. Are you surprised at the results?

45. Let $a$ and $b$ denote arbitrary scalars. Find $a\mathsf{u} + b\mathsf{v}$ given $\mathsf{u} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Then find the equation of the plane that consists of all such linear combinations of $\mathsf{u}$ and $\mathsf{v}$.

46. Find two vectors perpendicular to both $\mathsf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

47. Find a vector of length 5 that is perpendicular to both $\mathsf{u} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

48. Verify that $\|\mathsf{u} \times \mathsf{v}\| = \|\mathsf{u}\| \, \|\mathsf{v}\| \sin\theta$ with $\mathsf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

49. Find the equation of the plane passing through $A$, $B$, and $C$, given $A(2,1,3)$, $B(3,-1,5)$, $C(0,2,-4)$.

50. Find the equation of the plane containing $P(3,1,-1)$ and the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

51. Find equations of the form $ax + by = c$ for the lines in the Euclidean plane with the vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

52. Find vector equations for the line in the Euclidean plane whose equation is $y = 2x - 3$.

53. The intersection of two planes is, in general, a line.
    (a) Find three points on the line that is the intersection of the planes with equations $2x + y + z = 5$ and $x - y + z = 1$.
    (b) Find a vector equation for the line in 53(a).

54. Determine, with a simple reason, whether the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ and the plane with the equation $x - 3y + 2z = 4$ intersect. Find any points of intersection.

55. Let $\pi$ be the plane with equation $2x - 3y + 5z = 1$. Find a vector perpendicular to $\pi$.

56. Let $A(1,-1,-3)$, $B(2,0,-3)$, $C(2,-5,1)$, $D(3,-4,1)$ be the indicated points of 3-space.
    (a) Show that $A, B, C, D$ are the vertices of a parallelogram and find the area of this figure.
    (b) Find the area of triangle $ABC$.

57. Find the area of the triangle in the $xy$-plane with vertices $A(-2,3)$, $B(4,4)$, $C(5,-3)$.

58. Suppose a $2\times 2$ determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$. Show that one of the vectors $\begin{bmatrix} c \\ d \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ is a scalar multiple of the other. There are four cases to consider, depending on whether or not $a=0$ or $b=0$.

59. Determine whether or not the lines with equations $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 11 \\ -3 \\ -5 \end{bmatrix}$ intersect. Find any point of intersection.

60. Let $\ell_1$ and $\ell_2$ be the lines with the equations

$$\ell_1\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}; \qquad \ell_2\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ a \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ b \end{bmatrix}.$$

Given that these lines intersect at right angles, find the values of $a$ and $b$.

61. Find the equation of the plane each point of which is equidistant from the points $P(2,-1,3)$ and $Q(1,1,-1)$.

62. When does the projection of $\mathsf{u}$ on a nonzero vector $\mathsf{v}$ have a direction opposite that of $\mathsf{v}$?

63. Let $\mathsf{u} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$. Determine whether $\mathsf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in the plane spanned by $\mathsf{u}$ and $\mathsf{v}$.

64. In each case below, find the projection of $\mathsf{u}$ on $\mathsf{v}$ and the projection of $\mathsf{v}$ on $\mathsf{u}$.

    (a) $\mathsf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ (b) $\mathsf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

65. Find the distance from point $P(-1,2,1)$ to the line $\ell$ with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

66. Find the distance from $P(2,3,0)$ to the plane $\pi$ with equation $5x-y+z=1$. Then find the point of $\pi$ closest to $P$.

67. Find two orthogonal vectors that span each of the given planes:

(a) the plane spanned by $\mathsf{u} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(b) the plane with equation $2x - 3y + 4z = 0$

68. (a) Find two orthogonal vectors in the plane $\pi$ with equation $2x - y + z = 0$.

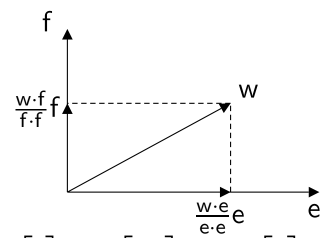
    (b) Use your answer to part (a) to find the projection of $\mathsf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on $\pi$.

69. (a) What is the projection of $\mathsf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ onto the plane $\pi$ with equation $2x + y - z = 0$?

    (b) What is the projection of $\mathsf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ onto $\pi$?

70. Find the projection of $\mathsf{w} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ on the plane $\pi$ whose equation is $x + y - 2z = 0$.

71. Suppose a vector $\mathsf{w}$ is a linear combination of nonzero orthogonal vectors $\mathsf{e}$ and $\mathsf{f}$. Prove that $\mathsf{w} = \dfrac{\mathsf{w} \cdot \mathsf{e}}{\mathsf{e} \cdot \mathsf{e}}\,\mathsf{e} + \dfrac{\mathsf{w} \cdot \mathsf{f}}{\mathsf{f} \cdot \mathsf{f}}\,\mathsf{f}$.




72. (a) The lines with vector equations $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$ are parallel. Why?

    (b) Find the distance between the lines in (a).

73. Most theorems in mathematics can be stated in the form "If $\mathcal{P}$, then $\mathcal{Q}$," symbolically $\mathcal{P} \implies \mathcal{Q}$. (Read "implies" for $\implies$.) The *converse* of this statement is $\mathcal{Q} \implies \mathcal{P}$: "If $\mathcal{Q}$, then $\mathcal{P}$." What is the converse of Theorem 5.2?

74. Don has a theory that to find the projection of a vector $\mathsf{w}$ on a plane $\pi$ (through the origin), one could project $\mathsf{w}$ on a normal $\mathsf{n}$ to the plane, and take $\operatorname{proj}_\pi \mathsf{w} = \mathsf{w} - \operatorname{proj}_{\mathsf{n}} \mathsf{w}$.

    (a) Is Don's theory correct? Explain by applying the definition of "projection onto a plane."

    (b) Illustrate Don's theory using the plane with equation $3x - 2y + z = 0$ and $\mathsf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

75. (The distance between skew lines.) Let $\ell_1$ and $\ell_2$ be the lines with equations

$$\ell_1\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \qquad \ell_2\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix}.$$

(a) Show that $\ell_1$ and $\ell_2$ are not parallel and that they do not intersect. (Such lines are called *skew*).

(b) Find the equation of the plane that contains $\ell_1$ and is parallel to $\ell_2$.

(c) Use the result of (b) to find the (shortest) distance between $\ell_1$ and $\ell_2$.

76. Let $\mathsf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$, $\mathsf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\mathsf{y} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$.

Find each of the following, if possible.

(a) $2\mathsf{x} - 3\mathsf{y} + \mathsf{u}$ (b) $\|\mathsf{y}\|$

77. Find $a$ and $b$, if possible, given $\begin{bmatrix} a-1 \\ 2b \\ 3 \\ a-b \end{bmatrix} = \begin{bmatrix} 1-b \\ -6 \\ -b \\ 8 \end{bmatrix}$.

78. Let $\mathsf{u} = \begin{bmatrix} -1 \\ a \\ b \\ 2 \\ 0 \end{bmatrix}$, $\mathsf{v} = \begin{bmatrix} c \\ 1 \\ 1 \\ a \\ -4 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, and $\mathsf{y} = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 4 \\ 8 \end{bmatrix}$.

Find $a$, $b$, and $c$, if possible, given $3\mathsf{u} - 2\mathsf{v} = \mathsf{y}$.

79. Find a unit vector that has the same direction as $\mathsf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

80. Let $\mathsf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Find all scalars $c$ with the property that $\|c\mathsf{u}\| = 8$.

81. Find the angle (to the nearest tenth of a radian and to the nearest degree) between $\mathsf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

82. Suppose $\mathsf{u}$ is a vector of length $\sqrt{6}$ and $\mathsf{v}$ is a unit vector orthogonal to $\mathsf{u} + \mathsf{v}$. Find $\|2\mathsf{u} - \mathsf{v}\|$.

83. Suppose $\mathsf{u}$, $\mathsf{v}$, $\mathsf{x}$, and $\mathsf{y}$ are unit vectors such that $\mathsf{v}\cdot\mathsf{x} = -\frac{1}{3}$, $\mathsf{u}$ and $\mathsf{y}$ are orthogonal, the angle between $\mathsf{v}$ and $\mathsf{y}$ is $45°$, and the angle between $\mathsf{u}$ and $\mathsf{x}$ is $60°$. Find $(\mathsf{u}+\mathsf{v})\cdot(\mathsf{x}-\mathsf{y})$.

84. Determine, with justification, whether $\mathsf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is a linear combination of $\mathsf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \\ 2 \end{bmatrix}$, $\mathsf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}$, $\mathsf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \\ -5 \end{bmatrix}$, $\mathsf{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, and $\mathsf{v}_5 = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

85. Verify the Cauchy–Schwarz and triangle inequalities for $\mathsf{u} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

86. Determine whether $\mathsf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\mathsf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, and $\mathsf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathsf{v}_4 = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ are linearly independent or linearly dependent.

87. Why have generations of linear algebra students refused to memorize the definitions of linear independence and linear dependence?

88. Suppose you were asked to prove that vectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_k$ are linearly independent. What would be the first line of your proof? What would be the last line?

89. Find $x$ and $y$, if possible, given $\begin{bmatrix} 2x-3y & -y \\ x-y & x+y \\ -x+y & x+2y \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & -1 \\ -3 & -3 \end{bmatrix}$.

90. Find $x$, $y$, $a$, and $b$ if possible, given $\begin{bmatrix} 2a+3x & 2 \\ -3 & y+b \end{bmatrix} = \begin{bmatrix} 0 & -2a-2x \\ a+x & 2 \end{bmatrix}$.

91. Let $A = \begin{bmatrix} 2 & x \\ y & 1 \end{bmatrix}$, $B = \begin{bmatrix} z & -4 \\ -1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 7 \\ 2 & -3 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 7 \\ 1 & -1 \end{bmatrix}$.

    Find $x$, $y$, and $z$, if possible, so that

    (a) $2A - B = C$ (b) $AB = C$

92. What is the $2 \times 3$ matrix $A$ for which $a_{ij} = j$?

93. Find $A+B$, $A-B$, and $2A-B$ given $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & 2 \end{bmatrix}$.

94. Find $AB$ and $BA$ (if defined) given $A = \begin{bmatrix} 1 & 2 & -4 \\ -6 & 8 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 \\ 2 & 6 \end{bmatrix}$.

95. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and $B = \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}$. What are the $(1,1)$ and $(2,2)$ entries of $AB$?

96. Write $3\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ in the form $A\mathsf{x}$, for a suitable matrix $A$ and vector $\mathsf{x}$. [This question tests understanding of the very important fact expressed in **6.3**.]

97. Express the system $\begin{aligned} 2x - \phantom{5}y &= 10 \\ 3x + 5y &= 7 \\ -x + \phantom{5}y &= -3 \\ 2x - 5y &= 1 \end{aligned}$ in the form $A\mathsf{x} = \mathsf{b}$.

98. The parabola with equation $y = ax^2 + bx + c$ passes through the points $(1,4)$ and $(2,8)$. Write a matrix equation whose solution is the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

99. Complete the following sentence:

    "A linear combination of the columns of a matrix $A$ is ____________."

100. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathsf{b} = \begin{bmatrix} 19 \\ 31 \\ 43 \end{bmatrix}$. Given that $A\mathsf{x} = \mathsf{b}$, write $\mathsf{b}$ as a linear combination of the columns of $A$.

101. Express $\mathsf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as a linear combination of the columns of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then find a vector $\mathsf{x}$ so that $A\mathsf{x} = \mathsf{b}$.

102. Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the $3 \times 3$ identity matrix. Express $\mathsf{x} = \begin{bmatrix} 20 \\ 35 \\ 47 \end{bmatrix}$ as a linear combination of the columns of $I$.

103. Suppose $\mathsf{x}_1$, $\mathsf{x}_2$, and $\mathsf{x}_3$ are three-dimensional vectors and $A$ is a $3 \times 3$ matrix such that $A\mathsf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\mathsf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A\mathsf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Let $X = \begin{bmatrix} \mathsf{x}_1 & \mathsf{x}_2 & \mathsf{x}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ be the matrix with columns $\mathsf{x}_1$, $\mathsf{x}_2$, $\mathsf{x}_3$. What is $AX$?

104. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Find $AD$ without calculation and explain your reasoning.

105. Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 & 4 \\ 4 & 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$.
Compute $AB$, $AC$, $AB + AC$, $B + C$, and $A(B + C)$.

106. Let $A$ be a square matrix. Does the equation $(A - 3I)(A + 2I) = 0$ imply $A = 3I$ or $A = -2I$? Answer by considering the matrix $A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$.

107. If $A$ is a matrix and $\mathsf{x}$ is a vector such that $\mathsf{x} + A\mathsf{x}$ is a vector, what can you say about the size of $A$ (and why)?

108. If $A$ and $B$ are $m \times n$ matrices and $A\mathsf{x} = B\mathsf{x}$ for each vector $\mathsf{x}$ in $\mathsf{R}^n$, show that $A = B$.

109. Let $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be nonzero and let $\mathsf{y}$ be any vector in $\mathsf{R}^m$. Prove that there exists an $m \times n$ matrix $A$ such that $A\mathsf{x} = \mathsf{y}$.

110. Determine whether or not $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 0 \end{bmatrix}$ and $B = \frac{1}{5}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & -2 \\ 5 & -1 & -1 \end{bmatrix}$ are inverses.

111. Given $A = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ find $A^{-1}$ and use this matrix to solve the system $\begin{aligned} -2x + 5y &= 3 \\ x - 3y &= 7. \end{aligned}$

112. Let $A = \begin{bmatrix} 1 & -7 & 1 \\ 2 & -9 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}$.

    (a) Compute $AB$ and $BA$.
    (b) Is $A$ invertible? Explain.

113. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$. Compute $A^T$, $B^T$, $AB$, $BA$, $(AB)^T$, $(BA)^T$, $A^TB^T$, and $B^TA^T$.

114. Verify Property 4 of the transpose of a matrix by showing that if $A$ is an invertible matrix, then so is $A^T$, and $(A^T)^{-1} = (A^{-1})^T$.

115. Suppose $A$ and $B$ are square matrices which commute. Prove that $(AB)^T = A^TB^T$.

116. Let $A$, $B$, and $C$ be matrices with $C$ invertible. If $AC = BC$, show that $A = B$.

117. Let $A$ be any $m \times n$ matrix. Let $\mathsf{x}$ be a vector in $\mathsf{R}^n$ and let $\mathsf{y}$ be a vector in $\mathsf{R}^m$. Explain the equation $\mathsf{y} \cdot (A\mathsf{x}) = \mathsf{y}^TA\mathsf{x}$.

118. Suppose $A$ and $B$ are invertible matrices and $X$ is a matrix such that $AXB = A+B$. What is $X$?

119. Is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in row echelon form?

120. Reduce each of the following matrices to row echelon form. In each case, identify the pivots and the pivot columns.

(a) $\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 4 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 3 & 13 & 20 & 8 \\ -2 & -10 & -16 & -4 \\ 1 & 10 & 38 & 28 \end{bmatrix}$

121. Repeat Exercise 120 for the matrix $\begin{bmatrix} 3 & -1 & 5 & 3 & 6 & -3 \\ 7 & -7 & 1 & 5 & 4 & 2 \\ -4 & 6 & 4 & -2 & 4 & 10 \\ 16 & -10 & 16 & 14 & 22 & -10 \\ -13 & 9 & -11 & -11 & -16 & 12 \end{bmatrix}$.

Identify the pivot columns, but this time don't go to the trouble of identifying the pivots.

122. Given $A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 6 & 6 \\ 6 & 6 & 10 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathsf{b} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, solve $A\mathsf{x} = \mathsf{b}$ expressing your solution as a vector or as a linear combination of vectors as appropriate.

123. Write the system $\begin{aligned} -2x_1 + \phantom{7}x_2 + \phantom{1}5x_3 &= -10 \\ -8x_1 + 7x_2 + 19x_3 &= -42 \end{aligned}$ in the form $A\mathsf{x} = \mathsf{b}$. What is $A$? What is $\mathsf{x}$? What is $\mathsf{b}$? Solve the system, expressing your answer as a vector or a linear combination of vectors as appropriate.

124. In solving the system $A\mathsf{x} = \mathsf{b}$, Gaussian elimination on the augmented matrix $[A|\mathsf{b}]$ led to the row echelon matrix $\left[\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 5 \end{array}\right]$. The variables were $x_1, x_2, x_3, x_4$.

(a) Circle the pivots. Identify the free variables.

(b) Write the solution to the given system as a vector or as a linear combination of vectors, as appropriate.

125. Here is a matrix in row echelon form: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

It represents the final row echelon matrix after Gaussian elimination was applied to the matrix $[A|\mathsf{b}]$ in an attempt to solve $A\mathsf{x} = \mathsf{b}$. If possible, find the solution (in vector form) of $A\mathsf{x} = \mathsf{b}$. If there is a solution, state whether this is unique or whether there are infinitely many solutions. The vector $\mathsf{x}$ is $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$.

126. Solve each of the following systems of linear equations by Gaussian elimination and back substitution. Write your answers as vectors or as linear combinations of vectors if appropriate.

(a) $$\begin{aligned} 2x - \phantom{2}y + 2z &= -4 \\ 3x + 2y \phantom{{}-6z} &= \phantom{-}1 \\ x + 3y - 6z &= \phantom{-}5 \end{aligned}$$

(b) $x - y + 2z = 4$

(c) $$\begin{aligned} 2x - \phantom{2}y + \phantom{6}z &= \phantom{-}2 \\ 3x + \phantom{2}y - 6z &= -9 \\ -x + 2y - 5z &= -4 \end{aligned}$$

(d) $$\begin{aligned} -6x_1 + \phantom{1}8x_2 - \phantom{1}5x_3 - 5x_4 &= -11 \\ -6x_1 + \phantom{1}7x_2 - 10x_3 - 8x_4 &= -9 \\ -8x_1 + 10x_2 - 10x_3 - 9x_4 &= -13 \end{aligned}$$

127. Determine whether $\begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}$ is a linear combination of the columns of $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 7 & 5 \\ 6 & -1 & 9 \end{bmatrix}$.

128. Determine whether $\begin{bmatrix} 2 \\ -11 \\ -3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$.

129. Let $\mathsf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\mathsf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathsf{v}_3 = -\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathsf{v}_4 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

Determine whether every vector in $\mathsf{R}^3$ is a linear combination of $\mathsf{v}_1$, $\mathsf{v}_2$, $\mathsf{v}_3$, and $\mathsf{v}_4$.

130. Determine whether or not the points $(-2,1,1)$, $(2,2,-2)$, $(4,7,-1)$, $(-10,11,-9)$ are *coplanar*, that is, lie in a plane. If the answer is "yes," give an equation of a plane on which the points all lie.

131. The planes with equations $x + 2z = 5$ and $2x + y = 2$ intersect in a line.

(a) Why?

(b) Find an equation of the line of intersection.

132. Suppose we want to find a quadratic polynomial $p(x) = a + bx + cx^2$ that passes through the points $(-2,3)$, $(0,-11)$, $(5,24)$.

(a) Write the system of equations that must be solved in order to determine $a$, $b$, and $c$.

(b) Find the desired polynomial.

133. Consider the system $\begin{aligned} 5x + 2y &= a \\ -15x - 6y &= b. \end{aligned}$

(a) Under what conditions (on $a$ and $b$), if any, does this system fail to have a solution?

(b) Under what conditions, if any, does the system have a unique solution? Find any unique solution that may exist.

(c) Under what conditions are there infinitely many solutions? Find these solutions if and when they exist.

134. (a) Find a condition on $a$, $b$, and $c$ that is both necessary and sufficient for the system

$$\begin{aligned} x - 2y &= a \\ -5x + 3y &= b \\ 3x + y &= c \end{aligned}$$

to have a solution.

(b) Could the system in part (a) have infinitely many solutions? Explain.

135. Solve $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 3x_2 \phantom{{}- x_3} &= 0 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$ by Gaussian elimination and back substitution. Write your answer as vectors or as linear combinations of vectors as appropriate.

136. (a) Express the zero vector $\mathsf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as a nontrivial linear combination of the columns of $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

(b) Use part (a) to show that $A$ is not invertible.

137. Find conditions on $a$, $b$, and $c$ (if any) such that the system

$$\begin{aligned} x + y \phantom{{}+ cz} &= 0 \\ y + z &= 0 \\ x \phantom{{}+ by} - z &= 0 \\ ax + by + cz &= 0 \end{aligned}$$

has

- a unique solution;

- no solution;
- an infinite number of solutions.

138. Write the solution to $\begin{aligned} x_1 + 5x_2 + 7x_3 &= -2 \\ 9x_3 &= \ \ 3 \end{aligned}$ in the form $\mathsf{x} = \mathsf{x}_p + \mathsf{x}_h$ where $\mathsf{x}_p$ is a particular solution and $\mathsf{x}_h$ is a solution of the corresponding homogeneous system.

139. Determine whether or not the vectors $\mathsf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $\mathsf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathsf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathsf{v}_4 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ are linearly independent. If they are not, write down a nontrivial linear combination of the vectors which equals the zero vector.

140. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and let $A$ be an arbitrary $3 \times n$ matrix, $n \geq 1$.

    (a) Explain the connection between $A$ and $EA$.

    (b) What is the name for a matrix such as $E$?

    (c) Is $E$ invertible? If so, what is its inverse?

141. Suppose $A$ is a $3 \times n$ matrix and we would like to modify $A$ by subtracting the second row from the third. Write the elementary matrix $E$ for which $EA$ is $A$ modified as specified.

142. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 8 & 9 \end{bmatrix}$. Write an elementary matrix $E$ so that $EA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & -4 & -9 \end{bmatrix}$.

143. Let $E$ be the $3 \times 3$ elementary matrix that adds 4 times row one to row two and $F$ the $3 \times 3$ elementary matrix that subtracts 3 times row two from row three.

    (a) Find $E$, $F$, $EF$, and $FE$ without calculation.

    (b) Write the inverses of the matrices in part (a) without calculation.

144. What number $x$ will force an interchange of rows when Gaussian elimination is applied to $\begin{bmatrix} 2 & 5 & 1 \\ 4 & x & 1 \\ 0 & 1 & -1 \end{bmatrix}$?

145. Let $A = \begin{bmatrix} \mathsf{a}_1 & \mathsf{a}_2 & \cdots & \mathsf{a}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ be an $m \times n$ matrix with columns the vectors $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n$ and let $B$ be the upper triangular matrix $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & b_{nn} \end{bmatrix}$. How are the columns

of $AB$ related to the columns of $B$?

146. Write the inverse of each of the following matrices without calculation. Explain your reasoning.

(a) $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

147. Find the inverse of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ simply by writing it down. No calculation is necessary, but explain your reasoning.

148. Show that the lower triangular matrix $L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ is invertible by finding a matrix $M$ with $LM = I$, the $2 \times 2$ identity matrix.

149. Let $A$ be a $3 \times n$ matrix, $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) How are the rows of $EA$, of $DA$ and of $PA$ related to the rows of $A$?

(b) Find $P^{-1}$.

150. Ian says $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ has no LU factorization but Lynn disagrees. Lynn adds the second row to the first, then subtracts the first from the second, like this

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = U.$$

She gets $L$ as the product of certain elementary matrices. Who is right? Explain.

151. Write $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$ as the product of elementary matrices (without calculation) and use this factorization to find $L^{-1}$.

152. Write $U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$ as the product of elementary matrices (without calculation) and use this factorization to find $U^{-1}$.

153. If possible, find an LU factorization of $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 1 & 7 \end{bmatrix}$ and express $L$ as the product of elementary matrices.

154. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$.

   (a) Find an elementary matrix $E$ and an upper triangular matrix $U$ such that $EA = U$.

   (b) Find a lower triangular matrix $L$ such that $A = LU$.

   (c) Express $A\mathsf{v}$ as a linear combination of the columns of $A$.

155. Let $A = \begin{bmatrix} -3 & 3 & 6 \\ 2 & 5 & 10 \\ 0 & 1 & 4 \end{bmatrix}$. Find an LU factorization of $A$ by moving $A$ to a row echelon matrix (with the elementary row operations), keeping track of the elementary matrices. Express $L$ as the product of elementary matrices.

156. Find, if possible, an LU factorization of each of the following matrices $A$:

   (a) $\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -6 & 5 \\ -4 & 12 & -9 \\ 2 & -9 & 8 \end{bmatrix}$

157. Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = LU$.

   (a) Use this factorization of $A$ to solve the system $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$.

   (b) Express $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$ as a linear combination of the columns of $A$.

158. Use the factorization $\begin{bmatrix} -2 & -6 \\ 1 & 15 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 12 \end{bmatrix}$ to solve $A\mathsf{x} = \mathsf{b}$ for $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, with $\mathsf{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, where $A = \begin{bmatrix} -2 & -6 \\ 1 & 15 \end{bmatrix}$.

159. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$ and $\mathsf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

   (a) Solve the system $A\mathsf{x} = \mathsf{b}$ by applying the method of Gaussian elimination to the augmented matrix.

   (b) Factor $A = LU$.

   (c) Why is $A$ invertible?

   (d) Find the first column of $A^{-1}$.

160. Solve $\begin{aligned} 2x_1 + 2x_2 &= 8 \\ 4x_1 + 9x_2 &= 21 \end{aligned}$ as follows:

i. Write the system in the form $A\mathsf{x} = \mathsf{b}$.

ii. Factor $A = LU$.

iii. Solve $L\mathsf{y} = \mathsf{b}$ for $\mathsf{y}$ by forward substitution and $U\mathsf{x} = \mathsf{y}$ by back substitution. Finally, write the solution to $A\mathsf{x} = \mathsf{b}$ in vector form.

161. Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ and $\mathsf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

i. find an LU factorization of $A$ and

ii. use this to solve $A\mathsf{x} = \mathsf{b}$.

162. Find a PLU factorization of $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

163. Suppose $A$ and $B$ are symmetric matrices. Is $AB$ symmetric (in general)?

164. A matrix $A$ is *skew-symmetric* if $A^T = -A$. Show that if $A$ and $B$ are $n \times n$ skew-symmetric matrices, then so is $A + B$.

165. True or false: The pivots of an invertible matrix $A$ are the nonzero diagonal entries of $U$ in an LU or a PLU factorization.

166. If $A$ is a symmetric $n \times n$ matrix and $\mathsf{x}$ is a vector in $\mathsf{R}^n$, show that $\mathsf{x}^T A^2 \mathsf{x} = \|A\mathsf{x}\|^2$.

167. Determine whether or not $A = \begin{bmatrix} -1 & 2 \\ -7 & 10 \end{bmatrix}$ and $B = \frac{1}{4}\begin{bmatrix} 10 & -2 \\ 7 & -1 \end{bmatrix}$ are inverses.

168. Find the reduced row echelon form of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ -3 & 3 & -5 \\ 2 & -1 & 4 \end{bmatrix}$.

169. Determine whether or not each of the following matrices has an inverse. Find the inverse whenever this exists.

(a) $\begin{bmatrix} 3 & 1 \\ -6 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

170. Consider the system of equations $\begin{aligned} x_1 - 2x_2 + 2x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 + x_3 &= -2. \end{aligned}$

i. Write this in the form $A\mathsf{x} = \mathsf{b}$.

ii. Solve the system by finding $A^{-1}$.

iii. Express $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

171. Given that $A$ is a $2 \times 2$ matrix and $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}^{-1} A \begin{bmatrix} 5 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find $A$.

172. Express $\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$ as the product of elementary matrices.

173. Use Theorem 8.16 to determine whether or not $\begin{bmatrix} -1 & 2 \\ 3 & -7 \end{bmatrix}$ is invertible.

174. Given two $n \times n$ matrices $X$ and $Y$, how do you determine whether or not $Y = X^{-1}$?

175. Is the factorization of an invertible matrix as the product of elementary matrices unique? Explain.

176. For each of the following matrices $A$,

   i. find the matrix $M$ of minors, find the matrix $C$ of cofactors, and compute the products $AC^T$ and $C^TA$;
   ii. find $\det A$;
   iii. if $A$ is invertible, find $A^{-1}$.

   (a) $A = \begin{bmatrix} 2 & -4 \\ -7 & 9 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 4 & 7 & 5 \end{bmatrix}$

177. Find the determinant of $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ expanding by cofactors of the third row.

178. Let $A = \begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 7 \\ -3 & 1 & 5 \end{bmatrix}$.

   (a) Find the matrix of minors.
   (b) Find cofactor matrix $C$.
   (c) Find the dot product of the second row of $A$ and the third row of $C$. Could this result have been anticipated? Explain.
   (d) Find the dot product of the first column of $A$ and the second column of $C$. Could this result have been anticipated? Explain.
   (e) Find the dot product of the second column of $A$ and the second column of $C$. What is the significance of this number?

179. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and suppose $C = \begin{bmatrix} -1 & -1 & c_{13} \\ c_{21} & 0 & -2 \\ -1 & c_{32} & 1 \end{bmatrix}$ is the matrix of cofactors of $A$.

   (a) Find the values of $c_{21}$, $c_{13}$ and $c_{32}$.

   (b) Find $\det A$.

   (c) Is $A$ invertible? Explain without further calculation.

   (d) Find $A^{-1}$, if this exists.

180. For what values of $x$ are the given matrices singular?

   (a) $\begin{bmatrix} x & 1 \\ -2 & x+3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & x \\ 0 & -3 & 0 \\ 4 & x & 7 \end{bmatrix}$

181. Professor G. was making up a homework problem for his class when he spilled a cup of coffee over his work. (This happens to Prof. G. all too frequently.) All that remained legible was this:

$$A = \begin{bmatrix} -1 & 2 & \\ 0 & 3 & \\ 2 & -2 & \end{bmatrix}; \quad \text{matrix of cofactors} = C = \begin{bmatrix} 19 & 10 & \\ -14 & -11 & \\ -2 & 5 & \end{bmatrix}.$$

   (a) Find the missing entries.

   (b) Find $A^{-1}$.

182. Find all values of $a$ which make $P(a) = \begin{bmatrix} 1 & 0 & 0 \\ a & 2 & 0 \\ 1 & 2a & a^2+a \end{bmatrix}$ singular.

183. If the matrix of cofactors of a matrix $A$ is $\begin{bmatrix} -1 & 7 \\ 2 & 4 \end{bmatrix}$, find $A$.

184. Given $\det A = 5$ and the matrix of cofactors $C = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, find the matrix $A$.

185. Given that $\begin{vmatrix} -1 & 7 & 3 & 4 \\ 0 & -5 & -3 & 2 \\ 6 & 1 & -1 & 3 \\ 2 & 2 & 0 & 1 \end{vmatrix} = 118$, find

(a) $\begin{vmatrix} -1 & 7 & 3 & 4 \\ 0 & -5 & -3 & 2 \\ 6 & 1 & -1 & 3 \\ -1 & 7 & 3 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 7 & 3 & 4 \\ 0 & -5 & -3 & 2 \\ 12 & 2 & -2 & 6 \\ 2 & 2 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 8 & 7 & 3 & 4 \\ 4 & -5 & -3 & 2 \\ 6 & 1 & -1 & 3 \\ 2 & 2 & 0 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} -1 & 7 & 3 & 4 \\ 6 & 1 & -1 & 3 \\ 0 & -5 & -3 & 2 \\ 2 & 2 & 0 & 1 \end{vmatrix}$

(e) $\begin{vmatrix} -1 & 0 & 6 & 2 \\ 7 & -5 & 1 & 2 \\ 3 & -3 & -1 & 0 \\ 4 & 2 & 3 & 1 \end{vmatrix}$

186. Let $A$ and $B$ be $5 \times 5$ matrices with $\det(-3A) = 4$ and $\det B^{-1} = 2$. Find $\det A$, $\det B$ and $\det AB$.

187. Let $A = \begin{bmatrix} 1 & 7 & 0 & 17 & 9 \\ 0 & 2 & -35 & 10 & 15 \\ 0 & 0 & 3 & -5 & 11 \\ 0 & 0 & 0 & 2 & 77 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$. Find $\det A$, $\det A^{-1}$, and $\det A^2$.

188. Let $A = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{bmatrix}$.

(a) Find the determinant of $A$ expanding by cofactors of the first column.

(b) Find the determinant of $A$ by reducing it to an upper triangular matrix.

189. Determine whether the vectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ are linearly independent or linearly dependent.

190. Find the determinants of each of the following matrices by reducing to upper triangular matrices.

(a) $\begin{bmatrix} -2 & 1 & 2 \\ 1 & 3 & 6 \\ -4 & 5 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \\ -6 & -2 & -4 & 2 \\ 1 & -1 & 0 & -1 \end{bmatrix}$

191. If $P$ is a permutation matrix and

$$PA = \begin{bmatrix} 1 & 6 & 7 \\ 5 & 25 & 36 \\ -2 & -27 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 7 \\ 0 & -5 & 1 \\ 0 & 0 & 7 \end{bmatrix} = LU,$$

what are the possible values for $\det A$?

192. Give at least three conditions on the rows of a matrix $A$ that guarantee that $A$ is singular.

193. Suppose each column of a $3 \times 3$ matrix $A$ is a linear combination of two vectors $\mathsf{u}$ and $\mathsf{v}$. Find an argument involving the determinant that shows that $A$ is singular.

194. Find the following determinants by methods of your choice.

(a) $\begin{vmatrix} 6 & -3 \\ 1 & -4 \end{vmatrix}$ (b) $\begin{vmatrix} 5 & 3 & 8 \\ -4 & 1 & 4 \\ -2 & 3 & 6 \end{vmatrix}$ (c) $\begin{vmatrix} -1 & 2 & 3 & 4 \\ 3 & -9 & 2 & 1 \\ 0 & -5 & 7 & 6 \\ 2 & -4 & -6 & -8 \end{vmatrix}$

195. Suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3$. Find $\begin{vmatrix} a & -g & 2d \\ b & -h & 2e \\ c & -i & 2f \end{vmatrix}$.

196. Suppose $A$ is a square matrix. Explain how an LDU factorization of $A$ could prove helpful in finding the determinant of $A$ and illustrate with $A = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 1 & -10 & -2 \\ 4 & 0 & -7 & 6 \\ 6 & -3 & 0 & 1 \end{bmatrix}$.

197. Suppose $A$ is an $n \times n$ invertible matrix and $A^{-1} = \frac{1}{2}(I - A)$, where $I$ is the $n \times n$ identity matrix. Show that $\det A$ is a power of 2.

198. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 5 & 0 \\ 2 & -1 & 4 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$. Is $\mathsf{v}$ an eigenvector for $A$? Explain and, if your answer is yes, give the corresponding eigenvalue.

199. Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, determine whether or not each of $\mathsf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathsf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathsf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathsf{v}_4 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathsf{v}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of $A$. Justify your answers and state the eigenvalues which correspond to any eigenvectors.

200. In each of the following situations, determine whether or not each of the given scalars is an eigenvalue of the given matrix. Justify your answers.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; $-1, 0, 1, 3, 5$ (b) $A = \begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$; $1, 2, 4, 5, 6$.

201. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

   (a) Show that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector of $A$ and state the corresponding eigenvalue.

   (b) What is the characteristic polynomial of $A$?

   (c) What are the eigenvalues of $A$?

202. Find the characteristic polynomial, the (real) eigenvalues, and the corresponding eigenspaces of each of the following matrices $A$.

   (a) $\begin{bmatrix} 5 & 8 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$

203. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, let $P$ be the point $(x, y)$ in the Cartesian plane, and let $\mathsf{v}$ be the vector $\overrightarrow{OP}$. Let $Q$ be the point such that $A\mathsf{v} = \overrightarrow{OQ}$.

   (a) Find the coordinates of $Q$.

   (b) Show that the line $\ell$ with equation $y = x$ is the right bisector of the line $PQ$ joining $P$ to $Q$; that is, $PQ$ is perpendicular to $\ell$ and the lines intersect at the midpoint of $PQ$.

   (c) Describe the action of multiplication of $A$ in geometrical terms.

   (d) Find the eigenvalues and corresponding eigenspaces of $A$ without calculation.

204. Multiplication by $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is reflection in a line. Find the equation of this line. Use the fact that the matrix is a reflection to find all eigenvectors and eigenvalues.

205. Let $\lambda$ be an eigenvalue of an $n \times n$ matrix $A$ and let $\mathsf{u}, \mathsf{v}$ be eigenvectors corresponding to $\lambda$. If $a$ and $b$ are scalars, show that $A(a\mathsf{u} + b\mathsf{v}) = \lambda(a\mathsf{u} + b\mathsf{v})$. Thus, if $a\mathsf{u} + b\mathsf{v} \neq \mathsf{0}$, it is also an eigenvector of $A$ corresponding to $\lambda$.

206. Suppose the components of each row of an $n \times n$ matrix $A$ add to 17. Prove that 17 is an eigenvalue of $A$.

207. Suppose $A$, $B$, and $P$ are $n \times n$ matrices with $P$ invertible and $PA = BP$. Show that every eigenvalue of $A$ is an eigenvalue of $B$.

208. What are the eigenvalues of $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$? What are the corresponding eigenvalues?

209. Let $\mathsf{v}$ be an eigenvector of a matrix $A$ with corresponding eigenvalue $\lambda$. Is $\mathsf{v}$ an eigenvector of $5A$? If so, what is the corresponding eigenvalue? Explain.

210. (a) Suppose a matrix $A$ is similar to the identity matrix. What can you say about $A$? Explain.

   (b) Can $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ be similar to the $2 \times 2$ identity matrix? Explain.

   (c) Is $A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$ diagonalizable? Explain.

211. Suppose $A$, $B$, and $C$ are $n \times n$ matrices. If $A$ is similar to $B$ and $B$ is similar to $C$, show that $A$ is similar to $C$.

212. Use the given fact that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ 22 & 8 \end{bmatrix}$ are similar matrices to find the determinant of $B$ and the characteristic polynomial of $B$.

213. (a) Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

   (b) Explain why the results of part (a) show that $A$ is not diagonalizable.

214. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

   (a) Show that $\lambda = 2$ is an eigenvalue of $A$ and find the corresponding eigenspace.

   (b) Given that the eigenvalues of $A$ are 2 and $-3$, what is the characteristic polynomial of $A$?

   (c) Why is $A$ diagonalizable?

   (d) Given that $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector of $A$ corresponding to $\lambda = -3$, find a matrix $P$ such that $P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

   (e) Let $Q = \begin{bmatrix} -2 & -6 \\ 4 & -3 \end{bmatrix}$. Find $Q^{-1}AQ$ without calculation.

215. Let $A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & -3 & -2 \\ 5 & 5 & 4 \end{bmatrix}$.

   (a) What is the characteristic polynomial of $A$?

   (b) Why is $A$ diagonalizable?

   (c) Find a matrix $P$ such that $P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

216. Determine whether or not each of the matrices $A$ given below is diagonalizable. If it is, find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. If it isn't, explain why.

(a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

217. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$.

(a) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

(b) Find a matrix $B$ such that $B^2 = A$. [Hint: First, find a matrix $D_1$ such that $D = D_1^2$.]

218. One of the most important and useful theorems in linear algebra says that a real symmetric matrix $A$ is always diagonalizable—in fact, *orthogonally diagonalizable*; that is, there is a matrix $P$ **with orthogonal columns** such that $P^{-1}AP$ is a diagonal matrix.) Illustrate this fact with respect to the matrix $A = \begin{bmatrix} 103 & -96 \\ -96 & 47 \end{bmatrix}$.

219. Repeat Exercise 218 for $A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}$.

220. Prove Theorem 12.12 in general; that is, if $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_k$ are eigenvectors of a matrix $A$ whose corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_k$ are linearly independent. [Hint: If the result is false, then there are scalars $c_1, c_2, \ldots, c_k$, not all 0, such that $c_1\mathsf{x}_1 + c_2\mathsf{x}_2 + \cdots + c_k\mathsf{x}_k = \mathsf{0}$. Omitting the terms with $c_i = 0$ and renaming the terms that remain, we may assume that

$$c_1\mathsf{x}_1 + c_2\mathsf{x}_2 + \cdots + c_\ell\mathsf{x}_\ell = \mathsf{0} \tag{†}$$

for some $\ell$, where $c_1 \neq 0$, $c_2 \neq 0$, $\ldots$, $c_\ell \neq 0$. Among all such equations, suppose the one exhibited is shortest; that is, whenever we have an equation like (†) (and no coefficients 0), there are at least $\ell$ terms. Note that $\ell > 1$, since otherwise $c_1\mathsf{x}_1 = \mathsf{0}$ and $\mathsf{x}_1 \neq \mathsf{0}$ gives $c_1 = 0$, a contradiction. Now imitate the proof of Theorem 12.12.]

221. Given that a matrix $A$ is similar to the matrix $\begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$, what's the characteristic polynomial of $A$? Explain.

222. If $A$ is a diagonalizable matrix with all eigenvalues nonnegative, show that $A$ has a square root; that is, $A = B^2$ for some matrix $B$. [Hint: Exercise 217.]