MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 9 **MATH 2050 sect. 3** ANSWERS

For practice and extra credit only.

1. Use vectors addition to show that

   (a) In a triangle ABC vector $\vec{AB} = -2\vec{NM}$, where M and N are midpoints of sides AC and BC respectively.
   Solution: $\vec{NM} = \vec{NC} + \vec{CM} = \frac{1}{2}\vec{BC} + \frac{1}{2}\vec{CA} = \frac{1}{2}\vec{BA} = -\frac{1}{2}\vec{AB}$. Thus $\vec{AB} = -2\vec{NM}$.

   (b) In a regular hexagon ABCDEF, $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = 3\vec{AD}$
   Solution: Note that $\vec{AE} = \vec{BD}$. Thus $\vec{AB} + \vec{AE} = \vec{AB} + \vec{BD} = \vec{AD}$. Similarly, since $\vec{AC} = \vec{FD}$, we have $\vec{AF} + \vec{AC} = \vec{AF} + \vec{FD} = \vec{AD}$. Finally,
   $$\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = (\vec{AB} + \vec{AE}) + (\vec{AF} + \vec{AC}) + \vec{AD} = 3\vec{AD}$$

   (c) In a regular n-gon $A_1 \cdots A_n$ with center O and even number of vertices the sum $\sum_{k=1}^{n} \vec{OA_k} = 0$.
   Solution: Let $n = 4$. We have a square $A_1A_2A_3A_4$ with the center O at the intersection of its diagonals. We observe that $\vec{OA_1} + \vec{OA_3} = \vec{0}$ and $\vec{OA_2} + \vec{OA_4} = \vec{0}$. Here $\vec{0}$ is a vector of zero magnitude.
   Similarly, for any even n there will be pairs of opposite vectors which cancel each other and thus the sum will always be zero.
   Is it true if the number of vertices is odd? Yes, it is true. To show that one can use either geometric arguments or physics (O is the center of mass), or complex numbers (n-th root of unity: $z^n = 1$ and $z \neq 1$ than $1 + z + \cdots + z^{n-1} = 0$). (Last two methods are beyond the scope of M2050 course.)

2. Find the intersection point of two lines with direction vectors (1,2) and (-1,3) respectively if it is known that the first line goes via point (0,0) and the second via point (1,1).
   Solution: Line one $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t$; Line two: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We need to solve the system $t = 1 - s$; $2t = 1 + 3s$. Find $s = 0.2$, $t = 0.8$. Thus intersection point has coordinates $x = 0.8$, $y = 1.6$.

3. Write equation of line going via points (1,2,3) and (4,5,6). Does point A belong to the line? If not, what is the distance from the point to the line?
   a) A(-1,0,1)
   b) A(5,7,9)
   Solution: Line has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} t$;

a) when $t = -2/3$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ thus this point belongs to the line.

b) point A(5,7,9) does not belong to the line (no value of t is possible). The distance from the point to the line is $\sqrt{2}$. It can be found from the right triangle with hypotenuse BA of magnitude $\sqrt{14}$ and leg BE of magnitude $2\sqrt{3}$. Here $B(4,5,6)$ is a point on the line and $\vec{BE}$ is the projection of $\vec{BA}$ onto the line.

4. Find the projection of vector $\vec{v} = (2,5,7)$ onto vector $\vec{u} = (-2,1,3)$.

   Solution: $\mathrm{Proj}_{\vec{u}}\vec{v} = \frac{11}{7}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$.

5. Consider parallelogram with vertices at

$$(1,1,1), \quad (3,6,8), \quad (-1,2,4), \quad (1,7,11)$$

   (a) find the area of the parallelogram

   (b) find equation of the plane to which the parallelogram belongs.

   Solution: The parallelogram's sides are vectors $\vec{u} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Their cross product is vector $\vec{u} \times \vec{v} = \begin{bmatrix} 8 \\ -20 \\ 12 \end{bmatrix} = 4\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$.

   The magnitude of $\vec{u} \times \vec{v}$ gives the area of the parallelogram. It is $4\sqrt{38}$.
   The equation of the plane is $2(x-1) - 5(y-1) + 3(z-1) = 0$, which can be simplified to $2x - 5y + 3z = 0$.

6. Give an example of two lines which intersect at point (1,2,3) at the right angle. Justify.

   Answer: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} s$.

   Both lines contain point (1,2,3) when $t = 0$ and $s = 0$. The dot product of the direction vectors is zero, thus the lines are orthogonal.

7. Give an example of a plane which is orthogonal to the vector (2,3,4). Explain.

   Answer: $2x + 3y + 4z = 5$. More general equation is $2(x - x_0) + 3(y - y_0) + 4(z - z_0) = 0$ Taking specific points (x_0, y_0, z_0) we obtain various possible constants. In my example the point was (1,1,0).