MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 8 **MATH 2050 sect. 3** ANSWERS.

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1. Find the characteristic polynomial, eigenvalues, eigenvectors and (if possible) an invertable matrix P such that $P^{-1}AP$ is diagonal.

   Hint: all eigenvalues in this problem are integers.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

   Answer:

   1) characteristic polynomial is $\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - 6 = \lambda^2 - 3\lambda - 4$

   2) eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$, and they are the roots of the characteristic equation $\lambda^2 - 3\lambda - 4 = 0$

   3) parametric solution of the system $(A - 4I)X = 0$ is $X = t\begin{bmatrix} 2 \\ 3 \end{bmatrix}$; parametric solution of the system $(A + I)X = 0$ is $X = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$;

   thus eigenvectors are $X_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   4) matrix $P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ so that $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ is diagonal.

   (b) $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

   Answer:

   1) characteristic polynomial is $\det(A - \lambda I)$
   $= (5 - \lambda)[(7 - \lambda)(-2 - \lambda) + 20] = (5 - \lambda)(\lambda^2 - 5\lambda + 6) = -(\lambda - 5)(\lambda - 3)(\lambda - 2)$

   2) eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 5$

   3) eigenvectors are $X_1 = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

   4) matrix $P = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix}$ so that $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is diagonal.

   (c) $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$

   Answer:

   1) characteristic polynomial is $\det(A - \lambda I) = (\lambda - 2)^3$.

   2) eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 2$ and $\lambda_3 = 2$

($\lambda = 2$ of multiplicity 3)
3) solution of the system $(A - 3I)X = 0$ has two parameters $X = sX_1 + tX_2$ which gives two eigenvectors $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$
4) since there only two eigenvectors there is no invertable matrix P and A is not diagonalizable.

(d) $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$

Answer:
1) characteristic polynomial is $\det(A - \lambda I) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$
2) eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$
3) eigenvectors are $X_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
4) matrix $P = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ so that $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonal.

2. Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that

a) the characteristic polynomial is $x^2 - trA \cdot x + \det A$ (recall $trA = a + d$).

Solution: characteristic polynomial is

$\det(A - xI) = (a - x)(d - x) - bc = x^2 - (a + d) + (ad - bc) = x^2 - trA \cdot x + \det A$.

b) the eigenvalues are

$$\frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$$

Solution: using formula for roots of quadratic equation $Ax^2 + Bx + C = 0$, with $A = 1$, $B = -(a + d)$, $C = ad - bc$ and after algebraic simplification we get this expression.

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{a+d}{2} + \sqrt{\frac{(a+d)^2 - 4ad + 4bc}{4}} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc.}$$

3. If $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, show that $AB = BA$.

Solution: Denote $D = P^{-1}AP$ and $E = P^{-1}BP$. Note that for any diagonal matrices $DE = ED$. Evaluate $DE = (P^{-1}AP)(P^{-1}BP) = P^{-1}ABP$, thus $AB = PDEP^{-1}$; Similarly, $ED = P^{-1}BAP$ and so $BA = PEDP^{-1} = PDEP^{-1} = AB$.

4. Suppose λ is an eigenvalue of a square matrix A. Show that λ^2 is an eigenvalue of A^2 (with the same eigenvector X).

   Solution: λ is an eigenvalue of a square matrix A means that for some vector X, $AX = \lambda X$.

   Consider $A^2X = AAX = A(AX) = A(\lambda X) = \lambda AX = \lambda^2 X$, which means that λ^2 is an eigenvalue of A^2.

   – What can you conjecture (and prove) about λ^k and A^k for any $k \geq 2$?

   Similarly, for any integer $k \geq 2$, λ^k is an eigenvalue of A^k (with the same eigenvector X).

5. Consider a linear dinamical sytem $V_{k+1} = AV_k$ for $k \geq 0$. Find exact formula for V_k. Approximate V_k for large values of k.

   (a) $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$, $V_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   Answer: $V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2$, where

   $\lambda_1 = 3$, $\lambda_2 = -2$ are eigenvalues of A,

   $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ are corresponding eigenvectors, and

   $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}V_0 = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 1/5 \end{bmatrix}$.

   Approximate V_k for large values of k is $V_k \approx \frac{6}{5}3^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}$ $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   Answer:

   $V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 + b_3\lambda_3^k X_3$, where

   $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 5$ are eigenvalues of A,

   $X_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are corresponding eigenvectors, and

   $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = P^{-1}V_0 = \frac{1}{28}\begin{bmatrix} -7 & 0 & 0 \\ 0 & -16 & 16 \\ 7 & 16 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 0 \\ 5/4 \end{bmatrix}$.

   Approximate V_k for large values of k is $V_k \approx \frac{5}{4}5^k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.