MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 **MATH 2050** ANSWERS

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1. (a) Give a definition of an eigenvalue and corresponding eigenvector of a matrix.

$$AX = \lambda X, \qquad X \neq 0.$$

(b) Explain the method (step by step) how to find all the eigenvalues and corresponding eigenvectors of a matrix.

1. Solve $\det(A - \lambda I) = 0$ for λ to find all eigen values.
2. For each value of λ found at step 1, solve homogeneous system $(A - \lambda I)X = 0$ for X to find corresponding eigenvector.
3. Check your eigen pair to satisfy the definition.

(c) Give an exapmle of a problem when knowing the eigenvalues and eigenvectors of a matrix can be useful to find the solution explicitly.

Knowing the eigenvalues λ_1, λ_2 and eigenvectors X_1, X_2 of a (2×2)-matrix A can be useful to find a power of this matrix, say A^{100} in a faster way:

$$A^{100} = PD^{100}P^{-1}, \quad D^{100} = \begin{bmatrix} \lambda_1^{100} & 0 \\ 0 & \lambda_2^{100} \end{bmatrix}, \quad P = [X_1 | X_2].$$

2. Find the characteristic polynomial, eigenvalues, eigenvectors and (if possible) an invertable matrix P such that $P^{-1}AP$ is diagonal. If the later is not possible, explain why.

(a) $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$

Answer:

characteristic polynomial: $\lambda^2 - 5\lambda - 14$;

eigenvalues, eigenvectors: $\lambda_1 = 7$, $X_1 = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$; $\lambda_2 = -2$, $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$;

matrix $P = \begin{bmatrix} 0.8 & -1 \\ 1 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$

Answer:

characteristic polynomial: $\lambda^2 - 3\lambda - 4$;

eigenvalues, eigenvectors: $\lambda_1 = 4$, $X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = -1$, $X_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$

matrix $P = \begin{bmatrix} 2 & -0.5 \\ 1 & 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

Answer:
characteristic polynomial: $\lambda^3 - 10\lambda^2 + 31\lambda - 30$;
eigenvalues, eigenvectors:

$$\lambda_1 = 2,\ X_1 = \begin{bmatrix} 0.8 \\ 0 \\ 1 \end{bmatrix};\ \lambda_2 = 3,\ X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix};\ \lambda_3 = 5,\ X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

matrix $P = \begin{bmatrix} 0.8 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$

Answer:
characteristic polynomial: $\lambda^3 - 6\lambda^2 + 11\lambda - 6$;
eigenvalues, eigenvectors:

$$\lambda_1 = 3,\ X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};\ \lambda_2 = 1, X_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}; \lambda_3 = 2,\ X_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix};$$

matrix $P = \begin{bmatrix} 0 & 1 & -1 \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

3. Let a matrix A have eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$ with corresponding eigenvectors $X_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Find the matrix A.

Answer: $A = PDP^{-1} = \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix}$. Find an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Answer: In general form $P = \begin{bmatrix} q & s & -t \\ 0 & -2s & -t \\ q & 3s & 3t \end{bmatrix}$ for any nonzero values t, s, q. For example,

$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & -1 \\ 1 & 3 & 3 \end{bmatrix}$ if parameters s, t, q all are set equal to 1.

5. Consider a linear dynamical system $V_{k+1} = AV_k$ for $k \geq 0$. Find exact formula for V_k. Approximate V_k for large values of k.

(a) $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$, $V_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Answer:

$$V_k = \frac{7^k}{9} \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \frac{13 \cdot (-2)^k}{9} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$V_k \approx \frac{7^k}{9} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ for large values of k.

(b) $A = \begin{bmatrix} -1 & 3 & -3 \\ -3 & 5 & -3 \\ -6 & 6 & -4 \end{bmatrix}$ $V_0 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

Answer: $V_k = 2^{k+1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

The eigenvalues in this problem are $\lambda_1 = -4$ and $\lambda_2 = \lambda_3 = 2$. That is, one of the eigenvalues is of multiplicity two.

Eigenvector for $\lambda_1 = -4$ is $X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda_2 = \lambda_3 = 2$ are $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

This gives $b_1 = 0$, $b_2 = 2$, $b_3 = 0$, and $V_k = b_2 \lambda_2^k X_2 = 2^{k+1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

6. True or False? Explain.

a) a) If λ is an eigenvalue of matrix A , then λ^{100} is an eigenvalue of the matrix A^{100}.

Answer: **True**. If $AX = \lambda X$, then $A^2 X = \lambda AX = \lambda^2 X$,and by induction, $A^{100}X = \lambda^{100} X$.

b) If we substitute matrix A into its characteristic polynomial then the result is the identity matrix.

Answer: **False**. The result is the zero matrix. This is the Hamilton-Cayley theorem, as discussed in class.

c) If $\lambda \neq 0$ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}.

Answer: **False**. The matrix A has to be invertible.