MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 7 **MATH 2050 sect. 3** ANSWERS.

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1. Evaluate determinant of each matrix by **two** ways:

   (1) using Laplace expansion, and

   (2) by reducing the matrix to the upper triangular form.

   (a) $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ *Answer:* $\det A = -1$.

   (b) $A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \\ -1 & -1 & 0 & 0 \end{bmatrix}$ *Answer:* $\det A = -11$.

   (c) $A = \begin{bmatrix} 1 & a & b \\ a & b & 1 \\ b & 1 & a \end{bmatrix}$ *Answer:* $\det A = 3ab - a^3 - b^3 - 1$.

   (d) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ *Answer:* $\det A = 0$.

2. The *characteristic polynomial* of matrix A is defined as $P_A(x) = \det(x \cdot I - A)$, where I is the identity matrix of the same size as A. Find the characteristic polynomial for $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$. Evaluate the polynomial for $x = A$.

   Solution

   $P_A(x) = \det(x \cdot I - A) = \det \begin{bmatrix} x-3 & -2 \\ -1 & x+1 \end{bmatrix} = x^2 - 2x - 5$.

   Then $P_A(A) = A^2 - 2A - 5I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

3. Let A, B, C be square matrices of the same size. Given $\det A = -1$, $\det B = 2$, and $\det C = 3$, find $\det A^5 B C^T A^{-1} B^2$.

   Answer: $\det A^5 B C^T A^{-1} B^2 = (-1)^5 \cdot 2 \cdot 3 \cdot (-1) \cdot 2^2 = 24$

4. Explain what can be said about $\det A$ if:

   (a) $A^2 = A$

   Solution: Let $A^2 = A$, then $\det(A^2) = \det A$. But $\det(A^2) = (\det A)^2$.
   Denote $x = detA$ and solve $x^2 = x$ for x. We get $x = 0$ or $x = 1$.
   Thus if $A^2 = A$ then $\det A$ can be either 0 or 1.

(b) $A^k = 0$ for some integer k. Here 0 is a zero matrix.

Solution: Let $A^k = 0$, then $\det A^k = 0$ for some k. This implies $(\det A)^k = 0$, and so $\det A = 0$.

(c) $A^2 + I = 0$

Solution: Let $A^2 = -I$, then $\det A^2 = \det(-I)$. There are two cases:

(1) the size of the matrices, n is even. Then $\det(-I) = 1$. The equation becomes $(\det A)^2 = 1$ thus $\det A$ can be either 1 or -1.

(2) the size of the matrices, n is odd. Then $\det(-I) = -1$. The equation becomes $(\det A)^2 = -1$ thus $\det A$ can't be real number. (it is an imaginary number either i or $-i$).

(d) $A^3 = A$

Solution: Let $A^3 = A$, then $\det(A^3) = \det A$, or equivaletly, $(\det A)^3 = \det A$. This implies that $\det A$ can be either 0, or 1 or -1.

(e) $A = A^T$

Solution: This condition implies that A is symmetric, but nothing more. Thus determinant of matrix A can have any number.

(f) $A^{-1} = A^T$

Solution: Let $A^{-1} = A^T$, then $\det(A^{-1}) = \det(A^T)$. Consequently, $\frac{1}{\det A} = \det A$. Thus $\det A^2 = 1$. We have: $\det A$ can be either 1 or -1.

5. Let a square matrix A was obtained from a square matrix B by a series of elementary row operations. Is it true that $\det A = \det B$? Give an example supporting your answer.

Answer. NO. If switching of two rows or multiplication of a row by a number was employed, the determinant will in general be changed.

If only third EROs were used, then the determinant will not change.

6. Find all values a such that the matrix is invertable

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & a \\ 2 & a & 1 \end{bmatrix}$

Solution:

$$\det \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & a \\ 2 & a & 1 \end{bmatrix} = a^2 - 4a + 3 = (a-3)(a-1).$$

The determinant becomes zeto at $a = 3$ or $a = 1$, and this gives values of a when the matrix is not invertable. For all other values of a this matrix is invertable.

(b) $\begin{bmatrix} 0 & a & -a \\ -1 & 1 & -1 \\ a & -a & a \end{bmatrix}$

Solution:

$$\det \begin{bmatrix} 0 & a & -a \\ -1 & 1 & -1 \\ a & -a & a \end{bmatrix} = 0$$

regardless of value of a.

Thus there is no such value a for which the matrix is invertable.