MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 5 **MATH 2050 sect. 3** ANSWERS

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1. Which of the following pairs of matrices are inverses of each other?

$$a) \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} -4 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Answer. To be inverses of each other two matrices must be of the same square size and give the identity matrix when multiplied in any order. Sinse the product BA is undefined these two matrices are not inverses of each other.

$$b) \qquad A = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} -3 & 6 & -3 \\ 6 & -21 & 12 \\ -3 & 14 & -8 \end{bmatrix}.$$

Answer. Here $AB = BA = I$, thus these two matrices are inverses of each other.

$$c) \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Answer. Here $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which is not the identity. Thus these two matrices are not inverses of each other.

2. Solve the system of equations by writing it in the form $AX = B$ and finding A^{-1}.

(a) $\begin{cases} 4x + 7y = 2 \\ x + 2y = -1 \end{cases}$

Solution. $A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$;

$A^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$; $X = A^{-1}B = \begin{bmatrix} 11 \\ -6 \end{bmatrix}$.

Answer: $x = 11$, $y = -6$.

(b) $\begin{cases} x - 2y + 2z = 3 \\ x + z = -2 \\ 2x + y + z = 0 \end{cases}$

Solution. $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$;

By a series of EROs we transform 3×6-matrix $[A|I]$ to $[I|*]$:

$$\left[\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -4 & 2 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & -1 & 5 & -2 \end{array}\right],$$

and find the inverse of A in place of $*$-block.

$A^{-1} = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 3 & -1 \\ -1 & 5 & -2 \end{bmatrix}$; then $X = A^{-1}B = \begin{bmatrix} 11 \\ -9 \\ -13 \end{bmatrix}$.

Answer: $x = 11$, $y = -9$, $z = -13$.

(c) $\begin{cases} y - z = 8 \\ x + 2y + z = 5 \\ x + z = -7 \end{cases}$

Solution. $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 5 \\ -7 \end{bmatrix}$;

$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$; $X = A^{-1}B = \begin{bmatrix} -5 \\ 6 \\ -2 \end{bmatrix}$.

Answer: $x = -5$, $y = 6$, $z = -2$.

3. Show that for any invertable square matrices A and B the following is true

$$((AB)^T)^{-1} = (A^T)^{-1}(B^T)^{-1}.$$

Solution. We use that $(AB)^T = B^T A^T$ and then $(CD)^{-1} = D^{-1}C^{-1}$ for any matrices such that all the operations are defined.

Thus,

$$((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1}.$$

4. Let A be a symmetric $n \times n$-matrix, and X, Y be matrices of the size $n \times 1$ and $1 \times n$ respectively. Show that

$$(YAX)^{-1} = (X^T A Y^T)^{-1}.$$

Solution One. Consider case $n = 2$. Take arbitrary symmetric 2×2-matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, and $X = \begin{bmatrix} d \\ e \end{bmatrix}$, $Y = \begin{bmatrix} f & h \end{bmatrix}$; then $X^T = \begin{bmatrix} d & e \end{bmatrix}$, $Y^T = \begin{bmatrix} f \\ h \end{bmatrix}$.

Calculate $YAX = dfa + dhb + feb + hec$. Similarly, $X^T A Y^T = dfa + dhb + feb + hec$, which is the same algebraic expresion. Thus their reciprocals $(YAX)^{-1}$ and $(X^T A Y^T)^{-1}$ are also equal (or both undefined if the number turned to be zero).

It remains to think hard and observe that since A is symmetric (equivalently, $A_{ij} = A_{ji}$) then for arbitrary n the number

$$YAX = \sum_{i=1}^{n}\sum_{j=1}^{n} Y_i A_{ij} X_j = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i A_{ij} Y_j = X^T A Y^T.$$

Thus, the reciprocals are equal.

Solution Two. Matrix A is symmetric, which means that $A = A^T$. Thus,

$$X^T A Y^T = X^T A^T Y^T = (YAX)^T = YAX.$$

The last equality is true since YAX is always 1×1 matrix, which is just a number; its transposition gives the number itself.

Again, since numbers YAX and $X^T A Y^T$ are equal, their reciprocals are as well.