1. Give an example of a system of linear equations with four variables and the rank as specified. Give a solution for your system and prove by substitution that your solution works for your system. Observe the relation between the rank, number of parameters and total number of the unknowns (variables).
(a) rank is 2 .

Answer:
The system is $\left\{\begin{array}{l}x+y+z+w=4 \\ w=1\end{array}\right.$. Then the solution is $\left\{\begin{array}{l}x=3-t-s \\ y=t \\ z=s \\ w=1\end{array}\right.$
There are two parameters in the solution.
(b) rank is 1 .

Answer:
The system is $\left\{x+y+z+w=4\right.$. Then the solution is $\left\{\begin{array}{l}x=4-t-s-q \\ y=t \\ z=s \\ w=q\end{array}\right.$.
There are 3 parameters in the solution.
(c) rank is 3 .

Answer:
The system is $\left\{\begin{array}{l}x+y+z+w=4 \\ z+w=0 \\ w=1\end{array}\right.$. Then the solution is $\left\{\begin{array}{l}x=4-t \\ y=t \\ z=-1 \\ w=1\end{array}\right.$.
There is one parameter in the solution.
(d) rank is 4 .

Answer:
The system is $\left\{\begin{array}{l}x+y+z+w=4 \\ y+z+w=3 \\ z+w=2 \\ w=1\end{array}\right.$. Then the solution is $\left\{\begin{array}{l}x=1 \\ y=1 \\ z=1 \\ w=1\end{array}\right.$.
(There are zero parameters in the solution.) Unique solution!
2. Find an equation of the plane passing through given points. Is the plane unique?
(a) plane $a x+b y+c z+d=0$ through ( $1,1,0$ ), $(5,0,1),(2,2,1)$

Answer: Unique plane $2 x+3 y-5 z=5$.

Solving the system $\left\{\begin{array}{l}a+b+d=0 \\ 5 a+c+d=0 \\ 2 a+2 b+c+d=0\end{array}\right.$ we obtain parametric solution with one parameter $\left\{\begin{array}{l}a=(-2 / 5) t \\ b=(-3 / 5) t \\ c=t \\ d=t\end{array}\right.$.
Take any non-zero value $t$, say $t=-5$ to get $2 x+3 y-5 z-5=0$. Note that there are infinitly many choices for $t$, but it is still the same plane as a geometrical object.
(b) plane $a x+b y+c z+d=0$ through ( $0,1,1$ ), ( $1,2,2$ ), ( $2,3,3$ )

Answer: Infinitly many planes, e.g. $y-z=0$.
Solving the system $\left\{\begin{array}{l}b+c+d=0 \\ a+2 b+2 c+d=0 \\ 2 a+3 b+3 c+d=0\end{array}\right.$ we obtain parametric solution with two parameters $\left\{\begin{array}{l}a=t \\ b=-s-t \\ c=s \\ d=t\end{array}\right.$. Take any values $s$ and $t$, say $t=0, s=-1$ to get $y-z=0$.
Note that for each choice of $s$ there are infinitly many choices for $t$, independent of $s$. Thus for each $s$ you get a plane; and sinse there are infinitly many choices for $s$, the number of different planes is infinite.
The true reason for that phenomena is that the three given points lie on the same line which makes the plane containing them be non-unique.
3. Find an equation of the curve passing through given points if it exists. Otherwise elain why it does not exist.
(a) line $a x+b y=c$ through points $(2,1),(1,2)$, and $(6,6)$;

Answer: no line.
Plot the point and you will see that they do not lie on the same line. Also, the system of linear equation has only trivial solution $a=b=c=0$. This implies the same conclusion.
(b) circle $x^{2}+y^{2}+a x+b y+c=0$ through $(7,-5),(5,-7),(5,-3)$.

Answer: $(x-5)^{2}+(y+5)^{2}=4$.
Plot the point and you will see that they belong to a circle with radius 2 and center at $(5,-5)$. Solve the system of linear equations to get $a=-10, b=10, c=46$. Complete the squares to get the answer.
(c) circle $x^{2}+y^{2}+a x+b y+c=0$ through $(0,1),(1,2),(2,3)$.

Answer: no such circle.
Plot the point and you will see that they lie on the same line. Also the system of linear equations for $a, b, c$ is inconsistent. No solution, thus no circle.
(d) parabola $a x^{2}+b x+c y+d=0$ through $(1,4),(4,13),(-2,1),(-5,4)$. Answer: $y=(1 / 3)(x+2)^{2}+1$ or $x^{2}+4 x-3 y+7=0$.
You will get a parametric solution with one parameter. Again, the algebraic answer is not unique(the equation may be multiplyed by any non-zero number) but the parabola is.
4. Find value $a$ such that the system has non-trivial solutions. Find the solutions.
(a) $\left\{\begin{array}{l}2 x+6 y+12 z=0 \\ x+2 y+z=0 \\ 2 x+3 y+a z=0\end{array}\right.$

Answer: Rewrite the system in REF.
The last equation of the system becomes: $(a+3) z=0$. We conclude that if $a=-3$ then $z=t$, and $y=-5 t, x=9 t$, where $t$ is any number(parameter). For any other value $a \neq-3, z=0$ and this implies only trivial solution.
(b) $\left\{\begin{array}{l}x-6 y+5 z=0 \\ x+a y-3 z=0 \\ -x+2 y-z=0\end{array}\right.$

Answer: Rewrite the system in REF.
If $a=2$ then $x=y=z=t$, where $t$ is any number.
5. True or False? Explain.
(a) Every homogeneous system of linear equations has a parametric solution. False
(b) Rank of a system of linear equations is always less than the number of variables. False
(c) If a system is inconsistent then it is definedly not homogeneous. True
(d) There is a circle through any three point in the $X Y$-plane. False
6. Compose your own True-or-False question. Answer is.

