entries are equal. Thus

$$\begin{aligned} 2x - 3y &= 8 \\ -y &= 2 \\ x - y &= 3 \\ x + y &= -1 \\ -x + y &= -3 \\ x + 2y &= -3. \end{aligned}$$

We find that $x = 1$, $y = -2$.

90. Equating $(2,1)$ entries gives $a + x = -3$. Equating $(1,2)$ entries gives $-2(a+x) = 2$, so $a + x = -1$. No x, y, a, b exist.

91. (a) Since $2A - B = \begin{bmatrix} 4 & 2x \\ 2y & 2 \end{bmatrix} - \begin{bmatrix} z & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 4-z & 2x+4 \\ 2y+1 & -3 \end{bmatrix}$, we must have $4 - z = 0$, $2x + 4 = 7$, and $2y + 1 = 2$, so $x = \frac{3}{2}$, $y = \frac{1}{2}$, and $z = 4$.

(b) Since $AB = \begin{bmatrix} 2z - x & -8 + 5x \\ yz - 1 & -4y + 5 \end{bmatrix}$, we need $2z - x = 0$, $-8 + 5x = 7$, $yz - 1 = 2$, and $-4y + 5 = -3$. The second equation says $5x = 15$, so $x = 3$. Then the first equation gives $z = \frac{1}{2}x = \frac{3}{2}$. The third equation says $yz = 3$, so $y = \frac{2}{3}(3) = 2$. Since $y = 2$ satisfies the last equation, we have a solution.

92. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

93. $A + B = \begin{bmatrix} 4 & -2 & 5 \\ -4 & 0 & 6 \end{bmatrix}$; $A - B = \begin{bmatrix} 0 & 4 & -3 \\ 2 & -2 & 2 \end{bmatrix}$; $2A - B = \begin{bmatrix} 2 & 5 & -2 \\ 1 & -3 & 6 \end{bmatrix}$.

94. AB is not defined; $BA = \begin{bmatrix} -27 & 26 & 16 \\ -34 & 52 & -2 \end{bmatrix}$.

95. The $(1,1)$ and $(2,2)$ entries of AB are, respectively, $ax + by + cz$ and $du + ev + fw$.

96. $3\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$.

97. $\begin{bmatrix} 2 & -1 \\ 3 & 5 \\ -1 & 1 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ -3 \\ 1 \end{bmatrix}$.

98. Substituting $x = 1$, $y = 4$ gives $a + b + c = 4$. Substituting $x = 2$, $y = 8$ gives $4a + 2b + c = 8$. These two equations correspond to the single matrix equation $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

99. A linear combination of the columns of a matrix A is $A\mathsf{x}$ for some vector x (whose components are the coefficients in the linear combination).

100. $\mathsf{b} = -4\begin{bmatrix}1\\4\\7\end{bmatrix} + \begin{bmatrix}2\\5\\8\end{bmatrix} + 7\begin{bmatrix}3\\6\\9\end{bmatrix}$.

101. We seek a and b so that $\begin{bmatrix}2\\3\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}$. We need $2 = a + b$ and $3 = b$, so $a = -1$. Thus $\begin{bmatrix}2\\3\end{bmatrix} = (-1)\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix}$; that is, $\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$.

102. $\mathsf{x} = 20\begin{bmatrix}1\\0\\0\end{bmatrix} + 35\begin{bmatrix}0\\1\\0\end{bmatrix} + 47\begin{bmatrix}0\\0\\1\end{bmatrix}$.

103. $AX = \begin{bmatrix}A\mathsf{x}_1 & A\mathsf{x}_2 & A\mathsf{x}_3\\ \downarrow & \downarrow & \downarrow\\ & & \end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

104. This is like Example 6.5. The first column of AD is A times the first column of D. By that important fact expressed in **6.3**, this is

$$-7 \times \text{ first column of } A + 0 \times \text{ second column of } A + 0 \times \text{ third column of } A,$$

which is $\begin{bmatrix}-7\\-7\\-7\end{bmatrix}$. The second and third columns of AD are obtained by similar means, giving $AD = \begin{bmatrix}-7&16&15\\-7&16&15\\-7&16&15\end{bmatrix}$.

105. $AB = \begin{bmatrix}8&5&0\\16&-5&20\end{bmatrix}$, $\quad AC = \begin{bmatrix}-3&-2&10\\-1&6&20\end{bmatrix}$, $\quad AB + AC = \begin{bmatrix}5&3&10\\15&1&40\end{bmatrix}$,
$B + C = \begin{bmatrix}1&-1&4\\3&1&7\end{bmatrix}$, $\quad A(B+C) = \begin{bmatrix}5&3&10\\15&1&40\end{bmatrix}$.

106. The answer is "no." With $A = \begin{bmatrix}3&-1\\0&-2\end{bmatrix}$, we find $A - 3I = \begin{bmatrix}0&-1\\0&-5\end{bmatrix}$ and $A + 2I = \begin{bmatrix}5&-1\\0&0\end{bmatrix}$, so that $(A - 3I)(A + 2I) = \begin{bmatrix}0&0\\0&0\end{bmatrix}$. Yet $A \neq 3I$ and $A \neq -2I$.

107. Let A be $m \times n$. Since $A\mathsf{x}$ exists, x is $n \times 1$ and $A\mathsf{x}$ is $m \times 1$. Thus $\mathsf{x} + A\mathsf{x}$ is the sum of a vector in R^n and a vector in R^m, so $m = n$. The matrix A is square.

108. We use the fact that $A\mathsf{e}_i$ is column i of A. (See **5.20**.) Since $A\mathsf{x} = B\mathsf{x}$ for all x, then certainly $A\mathsf{e}_1 = B\mathsf{e}_1$, so the first columns of A and B are the same. Similarly, $A\mathsf{e}_i = B\mathsf{e}_i$ implies that the ith columns of A and B are equal, for every i. Thus $A = B$ by **5.9**.

109. Since $\mathsf{x} \neq 0$, some component of x is not 0. Without loss of generality, $x_1 \neq 0$. [The reader should see how to adapt quickly our argument for any other x_i.] Let v be the vector $\frac{1}{x_1}\mathsf{y}$ and let $A = \begin{bmatrix} \mathsf{v} & 0 & \cdots & 0 \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$. By the fundamental **6.3**, $A\mathsf{x}$ is a linear combination of the columns of A, the coefficients being $x_1, x_2, \ldots, x_n$. Thus $A\mathsf{x} = x_1\mathsf{v} + x_2 0 + \cdots + x_n 0 = x_1\mathsf{v} = x_1(\frac{1}{x_1}\mathsf{y}) = \mathsf{y}$.

110. $AB = BA = I_3$, so these matrices are inverses.

111. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad-bc \neq 0$, then we saw in Example 8.8 that $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Here then, $A^{-1} = \begin{bmatrix} -3 & -5 \\ -1 & -2 \end{bmatrix}$. The system is $A\mathsf{x} = \mathsf{b}$ with $\mathsf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathsf{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, so the solution is $\mathsf{x} = A^{-1}\mathsf{b} = -\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -44 \\ -17 \end{bmatrix}$.

112. (a) $AB = \begin{bmatrix} 1 & -7 & 1 \\ 2 & -9 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

$$BA = \begin{bmatrix} -1 & 3 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -7 & 1 \\ 2 & -9 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -20 & 2 \\ 2 & -9 & 1 \\ 10 & -50 & 6 \end{bmatrix}.$$

(b) A is not invertible since $BA \neq I$. Only square matrices can be invertible.

113. $A^T = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, $B^T = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, $AB = \begin{bmatrix} 10 & -3 \\ 6 & -6 \end{bmatrix}$, $BA = \begin{bmatrix} 6 & 2 \\ 15 & -2 \end{bmatrix}$,

$(AB)^T = \begin{bmatrix} 10 & 6 \\ -3 & -6 \end{bmatrix}$, $(BA)^T = \begin{bmatrix} 6 & 15 \\ 2 & -2 \end{bmatrix}$, $A^TB^T = \begin{bmatrix} 6 & 15 \\ 2 & -2 \end{bmatrix}$, $B^TA^T = \begin{bmatrix} 10 & 6 \\ -3 & -6 \end{bmatrix}$.

114. We are asked to show that $(A^T)^{-1}$, which is the inverse of A^T, is the matrix $X = (A^{-1})^T$. Since both matrices are square, it suffices to prove that the product of A^T and X (in either order) is the identity matrix. Using the fact that $(BC)^T = C^TB^T$, we have $A^TX = A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$.

115. Since $AB = BA$, we have $(AB)^T = (BA)^T = A^TB^T$, as desired.

116. Multiplying $AC = BC$ on the right by C^{-1} gives $ACC^{-1} = BCC^{-1}$, so $AI = BI$ and $A = B$.

117. This follows immediately from **5.17**: $\mathsf{u} \cdot \mathsf{v} = \mathsf{u}^T\mathsf{v}$ for any vectors u and v.

118. Multiplying $AXB = A + B$ on the left by A^{-1} gives $XB = I + A^{-1}B$ and multiplying this on the right by B^{-1} gives $X = B^{-1} + A^{-1}$.

119. No, because the leading nonzero entries do not step to the right as you read down the matrix.