

Math-2320 Assignment 7 Solutions

Problem 1: (Section 7.1 Exercise 4) There are 30 people in a class learning about permutations. One after another, eight people gradually slip out the back door. In how many ways can this exodus occur?

Solution:

There are 30 choices for the first person to leave, 29 for the second, etc.
Thus this exodus can occur in

$$30 \times 29 \times 28 \times \cdots \times 23 = \frac{30!}{22!} = P(30, 8) \text{ ways.}$$

Problem 2: (Section 7.1 Exercise 7)

- (a.) In how many ways can ten boys and four girls sit in a row?
- (b.) In how many ways can they sit in a row if the boys are to sit together and the girls are to sit together?
- (c.) In how many ways can they sit in a row if the girls are to sit together?
- (d.) In how many ways can they sit in a row if **just** the girls are to sit together?

Solution:

- (a.) $14!$, which is the permutation of 14 people.
- (b.) $10! \cdot 4! \cdot 2$
- (c.) $4! \cdot 11!$ (treat the girls as one unit and count their possible permutations)
- (d.) $4! \cdot 11! - 10! \cdot 4! \cdot 2 = 9 \cdot 4! \cdot 10!$ (just girls sit together is the answer in (c.) minus the answer in (b.)).

Problem 3: (Section 7.1 Exercise 17)

- (a.) In how many numbers with seven distinct digits do only the digits 1 – 9 appear?
- (b.) How many of the numbers in (a.) contain a 3 and a 6?
- (c.) In how many of the numbers in (a.) do 3 and 6 occur consecutively (in either order)?
- (d.) How many of the numbers in (a.) contain neither a 3 nor a 6?
- (e.) How many of the numbers in (a.) contain a 3 but not a 6?
- (f.) In how many of the numbers in (a.) do exactly one of the numbers 3, 6 appear?
- (g.) In how many of the numbers in (a.) do neither of the consecutive pairs 36 and 63 appear?

Solution:

- (a.) $P(9, 7)$ which is $\frac{9!}{2!} = 9 \cdot 8 \cdot 7 \cdots 3 = 181,440$
- (b.) There are seven positions for 3, six positions for 6 and five positions for the remaining seven digits. Therefore, there are $7 \cdot 6 \cdot P(7, 5) = 105,840$ numbers which contain a 3

and a 6 in the numbers in (a.).

(c.) Treat 36 as a block. There are 6 positions for 36 and five positions for the remaining seven digits. Then all these for 63, which are $6 \cdot P(7, 5) \cdot 2 = 30240$.

(d.) $P(7, 7) = 5040$ (seven digits to be placed in seven positions.)

(e.) There are 7 positions for 3 and 6 positions for remaining 7 digits. Which gives, $7 \cdot P(7, 5) = 35,280$

(f.) Answer in (e.) times 2, which gives 70560.

(g.) Answer in (a.) minus answer in (c.), which is $181,440 - 30,240 = 151,200$.

Problem 4: (Section 7.2 Exercise 2) In how many ways can 12 players be divided into two teams of six for a game of street hockey?

Solution:

$$C(12, 6) = 924$$

Problem 5: (Section 7.2 Exercise 7) In a popular lottery known as "Lotto 6/49", a player marks a card with six different numbers from the integers 1 – 49 and wins if his or her numbers match six randomly selected such numbers.

(a.) In how many ways can a player complete a game card?

(b.) How many cards should you complete in order to have at least one chance in a million of winning?

(c.) In how many ways complete a game card so that no number matches any of the six selected?

(d.) In how many ways can a player complete a game card so that at least one number matches one of the six selected?

Solution:

(a.) $C(49, 6) = 13,983,816$

(b.) $\frac{n}{13,983,816} = \frac{1}{1,000,000}$. Therefore, $n = \lceil 13.9 \rceil = 14$

(c.) $C(49 - 6, 6) = C(43, 6) = 6,096,454$

(d.) Answer in (a.) minus the answer in (c.), which gives $= 13,983,816 - 6,096,454 = 7,887,362$

Problem 6: (Section 7.2 Exercise 16) The Head of the Department of Mathematical Sciences at a certain university has 12 mathematicians, seven computer scientists, and three

statisticians in his employ. He wishes to appoint some committees from among these 22 people.

- (a.) How many five-member committees can he appoint?
- (b.) How many five-member committees, each containing at least one statistician, can he appoint?
- (c.) A certain professor of mathematics, Dr.G, and a certain colleague, Dr.P, refuse to serve together on the same committee. How many five-member committees can be formed so as not to contain both Dr.G and Dr.P?
- (d.) How many five-member committees can be formed so that the number of mathematicians is greater than the number of computer scientists and the number of computer scientists is greater than the number of statisticians?

Solution:

- (a.) $C(22, 5) = 26,334$
- (b.) The number of committees with no statisticians is $C(22 - 3, 5) = C(19, 5)$. Therefore, at least one statistician is in $C(22, 5) - C(19, 5) = 14,706$
- (c.) $C(22, 5) - C(20, 3) = 25,194$. Which is the total number of committees minus number of committees in which both Dr.G and Dr.P are participants.
- (d.) The total members is 5 thus it must be 3 mathematicians, 2 computer science, no statisticians.
OR 4 mathematicians , 1 computer science and no statisticians. $C(12, 3) \cdot C(7, 2) + C(12, 4) \cdot C(7, 1) = 8085$

Problem 7: (Section 7.3 Exercise 15(b),(d)) Find the number of arrangements of the letters of each of the following words:

- (b.) SASKATOON
- (d.) CINCINNATI

Solution:

- (b.) $\frac{9!}{2! \cdot 2! \cdot 2!} = 45,360$ (Letters S, A, O are presented twice).
- (d.) $\frac{10!}{2! \cdot 3! \cdot 3!} = 50,400$ (Letter C appears twice, I, N appears 3 times)

Problem 8: (Section 7.3 Exercise 18)

- (a.) Show that there is a one-to-one correspondence between the number of ways to put ten identical marbles into three boxes and the number of ordered triples (x, y, z) of nonnegative integers which satisfy $x + y + z = 10$

- (b.) How many triples (x, y, z) of nonnegative integers satisfy $x + y + z = 10$?
 (c.) How many 5-tuples (x, y, z, u, v) of nonnegative integers satisfy $x + y + z + u + v = 19$?

Solution:

(a.) x is the number of marbles in box 1

y is the number of marbles in box 2

z is the number of marbles in box 3

Then $x + y + z = 10$, $x \geq 0$, $y \geq 0$, $z \geq 0$ each such triple represents a way to put 10 marbles in 3 boxes.

(b.) There are $C(10+2, 2) = 66$ triples of nonnegative integers which satisfy $x+y+z = 10$.

(c.) There are $C(19 + 4, 4) = 8855$ ways to put 19 identical marbles into 5 boxes.

Problem 9: (Section 7.5 Exercise 3(c)) For each of the following, expand using the Binomial Theorem and simplify.

(c.) $(2x^3 - \frac{1}{y^2})^8$

Solution:

Using the Binomial Theorem you obtain the following:

$$\begin{aligned} (2x^3 - \frac{1}{x^2})^8 &= C(8, 0) \cdot (2x^3)^8 + \\ &+ C(8, 1)(2x^3)^7 \cdot (\frac{-1}{x^2}) + C(8, 2)(2x^3)^6(\frac{-1}{x^2})^2 + C(8, 3)(2x^3)^5(\frac{-1}{x^2})^3 + C(8, 4)(2x^3)^4(\frac{-1}{x^2})^4 + \\ &+ C(8, 5)(2x^3)^3(\frac{-1}{x^2})^5 + C(8, 6)(2x^3)^2(\frac{-1}{x^2})^6 + C(8, 7)(2x^3)(\frac{-1}{x^2})^7 + C(8, 8)(\frac{-1}{x^2})^8 = \\ &= 256x^{24} - 1024x^{19} + 1729x^{14} - 1792x^9 + 1120x^4 - \frac{448}{x} + \frac{112}{x^6} - \frac{16}{x^{11}} + \frac{1}{x^{16}} \end{aligned}$$

Problem 10: (Section 7.5 Exercise 5) Consider the binomial expansion of $(x + y)^{20}$.

- (a.) What are the first three terms?
 (b.) What are the last three terms?
 (c.) What is the seventh term? The fifteenth term?
 (d.) What is the coefficient of $x^{13}y^7$?

Solution:

(a.) The first 3 terms are: x^{20} , $20x^{19}y$, $190x^{18}y^2$.

(b.) The last 3 terms are: $190x^2y^{18}$, $20xy^{19}$, y^{20} .

(c.) The seventh term is: $C(20, 6)x^{14}y^6$. The fifteenth term is: $C(20, 14)x^6y^{14}$.

(d.) $C(20, 7) = C(20, 13) = 77, 520$

Problem 11: (Section 7.5 Exercise 12) Find the coefficient of x^{25} in the binomial expansion of $(2x - \frac{3}{x^2})^{58}$

Solution:

General term of the expansion is

$$C(58, k) \cdot (2x)^{58-k} \cdot \left(\frac{-3}{x^2}\right)^k = C(58, k) \cdot 2^{58-k} \cdot (-3)^k \cdot x^{58-k-2k}$$

Therefore, we set $58 - 3k = 25$, so $k = 11$. So the coefficient is $C(58, 11) \cdot (2)^{47} \cdot (-3)^{11}$.

Problem 12: (Section 5.3 Exercise 2) Solve the recurrence relation $a_n = -6a_{n-1} + 7a_{n-2}$, $n \geq 2$, given $a_0 = 32$, $a_1 = -17$.

Solution:

The characteristic equation is $x^2 + 6x - 7$. Therefore roots are: $x_1 = 7$ and $x_2 = 1$

$$a_n = C_1(-7)^n + C_2(1^n)$$

$$a_0 = C_1 + C_2 = 32$$

$$a_1 = C_1(-7) + C_2 = -17$$

Therefore, $C_1 = \frac{49}{8}$, $C_2 = \frac{207}{8}$ and $a_n = \frac{49}{8}(-7)^n + \frac{207}{8}(1)^n$

Problem 13: (Section 5.3 Exercise 4) Solve the recurrence relation $a_{n+1} = 7a_n - 10a_{n-1}$, $n \geq 2$, given $a_1 = 10$, $a_2 = 29$.

Solution:

The characteristic equation is $x^2 - 7x + 10$. Therefore roots are: $x_1 = 2$ and $x_2 = 5$

$$a_n = C_1(2)^n + C_2(5)^n$$

$$a_1 = C_1 \cdot 2 + C_2 \cdot 5 = 10$$

$$a_2 = C_1 \cdot 4 + C_2 \cdot 25 = 29$$

Therefore, $C_1 = \frac{7}{2}$, $C_2 = \frac{3}{5}$ and $a_n = 7 \cdot 2^{n-1} + 3 \cdot 5^{n-1}$

Problem 14: (Section 5.3 Exercise 8) Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$, $n \geq 2$, given $a_0 = 40$, $a_1 = 37$.

Solution:

The characteristic equation is $x^2 - 2x + 1$. Therefore roots are: $x = 1$ (double root)

$$a_n = C_1(1)^n + n \cdot C_2(1)^n$$

$$a_0 = C_1 = 40$$

$$a_1 = C_1 + C_2 = 37$$

Therefore, $a_n = 40 - 3n$.