## M2320-Assignment 6: Solutions

Problem 1: (Section 6.1 Exercise 14) How many integers between 1 and 1000 (exclusive) are not divisible by $2,3,5$, or 7 ?

## Solution:

(a.)

Let $A, B, C$, and $D$ be the sets of integers between 1 and 1000 (inclusive) which are divisible by 2 , by 3 , by 5 , and by 7 , respectively. We want $\left|(A \cup B \cup C \cup D)^{c}\right|$. Now
$|A \cup B \cup C \cup D|=|A|+|B|+|C|+|D|-|A \cap B|-|A \cap C|-|A \cap D|-|B \cap C|-\mid B \cap$ $D|-|C \cap D|+|A \cap B \cap C|+|A \cap B \cap D|+|A \cap C \cap D|+|B \cap C \cap D|-|A \cap B \cap C \cap D|$

We have
$|A|=\left\lfloor\frac{1000}{2}\right\rfloor=500 \quad|B|=\left\lfloor\frac{1000}{3}\right\rfloor=333 \quad|C|=\left\lfloor\frac{1000}{5}\right\rfloor=200 \quad|D|=\left\lfloor\frac{1000}{7}\right\rfloor=142$
$|A \cap B|=\left\lfloor\frac{1000}{6}\right\rfloor=166 \quad|A \cap C|=\left\lfloor\frac{1000}{10}\right\rfloor=100 \quad|A \cap D|=\left\lfloor\frac{1000}{14}\right\rfloor=71$
$|B \cap C|=\left\lfloor\frac{1000}{15}\right\rfloor=66 \quad|B \cap D|=\left\lfloor\frac{1000}{21}\right\rfloor=47 \quad|C \cap D|=\left\lfloor\frac{1000}{35}\right\rfloor=28$
$|A \cap B \cap C|=\left\lfloor\frac{1000}{30}\right\rfloor=33 \quad|A \cap B \cap D|=\left\lfloor\frac{1000}{42}\right\rfloor=23$
$|A \cap C \cap D|=\left\lfloor\frac{1000}{70}\right\rfloor=14 \quad|B \cap C \cap D|=\left\lfloor\frac{1000}{105}\right\rfloor=9$

$$
|A \cap B \cap C \cap D|=\left\lfloor\frac{1000}{210}\right\rfloor=4
$$

So $|A \cup B \cup C \cup D|=772$. Our answer is $1000-772=228$.
(b.)

Now 1 and 1000 must be excluded. Since 1 is not divisible by 2, 3, 4 or 7 while 1000 is divisible by 5 , the answer is $228-1=227$.

Problem 2: (Section 6.1 Exercise 15) Find the number of integers between 1 and 10, 000 inclusive which are:
(a.) divisible by at least one of $3,5,7,11$;
(b.) divisible by 3 and 5 , but not by either 7 or 11;
(c.) divisible by exactly three of $3,5,7,11$;
(d.) divisible by at most three of $3,5,7,11$;

## Solution:

Let $A, B, C$, and $D$ be the sets of integers between 1 and 10,000 (inclusive) which are divisible by 3 , by 5 , by 7 , and by 11 respectively. We have
$|A|=\left\lfloor\frac{10000}{3}\right\rfloor=3333 \quad|B|=\left\lfloor\frac{10000}{5}\right\rfloor=2000 \quad|C|=\left\lfloor\frac{10000}{7}\right\rfloor=1428 \quad|D|=\left\lfloor\frac{10000}{11}\right\rfloor=$ 909

$$
\begin{gathered}
|A \cap B|=\left\lfloor\frac{10000}{15}\right\rfloor=666 \quad|A \cap C|=\left\lfloor\frac{10000}{21}\right\rfloor=476 \quad|A \cap D|=\left\lfloor\frac{10000}{33}\right\rfloor=303 \\
|B \cap C|=\left\lfloor\frac{10000}{35}\right\rfloor=285 \quad|B \cap D|=\left\lfloor\frac{10000}{55}\right\rfloor=181 \quad|C \cap D|=\left\lfloor\frac{10000}{77}\right\rfloor=129 \\
|A \cap B \cap C|=\left\lfloor\frac{10000}{105}\right\rfloor=95 \quad|A \cap B \cap D|=\left\lfloor\frac{10000}{165}\right\rfloor=60 \\
|A \cap C \cap D|=\left\lfloor\frac{10000}{231}\right\rfloor=43 \quad|B \cap C \cap D|=\left\lfloor\frac{10000}{385}\right\rfloor=25 \\
|A \cap B \cap C \cap D|=\left\lfloor\frac{10000}{1155}\right\rfloor=8
\end{gathered}
$$

(a.) This part asks us for $|A \cup B \cup C \cup D|$ we have
$|A \cup B \cup C \cup D|=|A|+|B|+|C|+|D|-|A \cap B|-|A \cap C|-|A \cap D|-|B \cap C|-\mid B \cap$ $D|-|C \cap D|+|A \cap B \cap C|+|A \cap B \cap D|+|A \cap C \cap D|+|B \cap C \cap D|-|A \cap B \cap C \cap D|$

$$
\begin{gathered}
=3333+2000+1428+909-666-476-303-285-181-129+95+60+43+25-8 \\
=5845
\end{gathered}
$$

(b.) This part asks for $|(A \cap B) \backslash(C \cup D)|=|A \cap B|-|(A \cap B) \cap(C \cup D)|$. Now $(A \cap B) \cap(C \cup D)=(A \cap B \cap C) \cup(A \cap B \cap D)$, so

$$
\begin{aligned}
& |(A \cap B) \cap(C \cup D)|=|A \cap B \cap C|+|(A \cap B \cap D)|-|(A \cap B \cap C) \cap(A \cap B \cap D)| \\
& \begin{array}{c}
=|A \cap B \cap C|+|(A \cap B \cap D)|-|(A \cap B \cap C \cap D)| \\
=95+60-8 \\
=147
\end{array}
\end{aligned}
$$

The answer is $666-147=519$.
(c.) This question asks for the number of elements in
$((A \cap B \cap C) \backslash D) \cup((A \cap B \cap D) \backslash C) \cup((A \cap C \cap D) \backslash B) \cup((B \cap C \cap D) \backslash A)$.
Since these sets are pairwise disjoint, that is the intersection of two of the four is empty, the number of elements in the union is the sum of the elements in each set. Since

$$
\begin{aligned}
& |(A \cap B \cap C) \backslash D|=|A \cap B \cap C|-|A \cap B \cap C \cap D|=95-8=87 \\
& |(A \cap B \cap D) \backslash C|=|A \cap B \cap d|-|A \cap B \cap C \cap D|=60-8=52 \\
& |(A \cap C \cap D) \backslash B|=|A \cap C \cap D|-|A \cap B \cap C \cap D|=43-8=35 \\
& |(B \cap C \cap D) \backslash A|=|B \cap C \cap D|-|A \cap B \cap C \cap D|=25-8=17
\end{aligned}
$$

The answer is $87+52+35+17=191$.
(d.) This part asks for the number of elements in $(A \cap B \cap C \cap D)^{C}$, which is $10000-$ $|A \cap B \cap C \cap D|=10000-8=9992$.

Problem 3: (Section 6.1 Exercise 19) Suppose $A$ and $B$ are finite sets with $|A \cup B|<$ $|A|+|B|$. Show that $A$ and $B$ have an element in common.

## Solution:

Let $A$ and $B$ do not have an element in common. This would mean that the intersection $A \cap B=\emptyset$ (empty set). Thus $|A \cap B|=0$. But then $|A \cup B|=|A|+|B|-|A \cap B|=|A|+|B|$. This contradicts with $|A \cup B|<|A|+|B|$. Thus our assumption about $A$ and $B$ not having common element was wrong. So $A$ and $B$ do have a common element.

Problem 4: (Section 6.1 Exercise 20) Suppose $A$ and $B$ are finite sets. Prove that $|A \times B|=|A| \times|B|$.

## Solution:

By definition of Cartesian product

$$
A \times B=\{(a, b) \mid a \in A, b \in B\}
$$

Let set $A$ contain $n$ elements, that is $|A|=n$. Let set $B$ contain $m$ elements, that is $|B|=m$. Then, by Product Rule, set of pairs $A \times B$ contains $n m$ elements. Thus $|A \times B|=|A| \times|B|$.

Problem 5: (Section 6.2 Exercise 8) In how many ways can one draw from a standard deck of 52 playing cards:
(a.) a heart or a spade?
(b.) an ace or a king?
(c.) a card numbered 2 through 10 ?
(d.) a card numbered 2 through 10 or a king?

## Solution:

(a.) 26
(b.) 8
(c.) 36
(d.) 40

Problem 6: (section 6.2 Exercise 10) How many possible telephone numbers consist of seven digits, the first two in the range of $2-9$ (inclusive), the third in the range of $1-9$ (inclusive), and each of the last four in the range $0-9$ (inclusive)?

## Solution:

$$
8 \times 8 \times 9 \times 10 \times 10 \times 10 \times 10=5,760,000
$$

Problem 7: (Section 6.2 Exercise 17) Two dice are rolled: (a.) In how many ways can a total of eight arise?
(b.) In how many ways can a total of seven arise?
(c.) In how many ways can a total of eight of seven arise?
(d.) In how many ways can one get "doubles" (the dice land with the same side up)?

## Solution:

(a.) Here are the possibilities. There are five ways to get a total of eight.

| First Die | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Second Die | 6 | 5 | 4 | 3 | 2 |

(b.) Here are the possibilities. There are six ways to get a total of seven.

| First Die | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Second Die | 6 | 5 | 4 | 3 | 2 | 1 |

(c.) There are $5+6=11$ ways to get an eight or a seven.
(d.) Getting doubles means two $1^{\prime} s$, two $2^{\prime} s$, two $3^{\prime} s$, two $4^{\prime} s$, two $5^{\prime} s$ or two $6^{\prime} s$. There are six ways to get doubles.

Problem 8: (Section 6.2 Exercise 22) How many possible license plates can be manufactured if a license plate consists of three letters followed by three digits and
(a.) the digits must be distinct: the letters can be arbitrary?
(b.) the letters must be distinct; the digits can be arbitrary?
(c.) the digits and the letters must be distinct?

## Solution:

(a.) $26 \times 26 \times 26 \times 10 \times 9 \times 8=12,654,720$
(b.) $26 \times 25 \times 24 \times 10 \times 10 \times 10=15,600,000$
(c.) $26 \times 25 \times 24 \times 10 \times 9 \times 8=11,232,000$

Problem 9: (Section 6.3 Exercise 8) In a gathering of 30 people, there are 104 different pairs of people who know each other.
(a.) Show that some person must have at least seven acquaintances.
(b.) Show that some person must have fewer than seven acquaintances.

## Solution:

(a.) Make a box for each person and put into that box a checkmark for each acquaintance with that person. There are 208 checkmarks to be entered into these boxes, so some must contain at least $\left\lceil\frac{208}{30}\right\rceil=7$ checkmarks.
(b.) If this is wrong, then each person knows at least seven others. This would mean at least $30(7)=210$ ordered pairs of people who know each other; that is, in the 210 , Herb and Edgar are counted twice, once because Herb knows Edgar and once because Edgar knows Herd. So we get at least $\frac{1}{2}(210)=105$ pairs of acquaintances, a contradiction.

Problem 10: (Section 6.3 Exercise 12) George has promised to increase his vocabulary by learning the meanings of 90 new words during his summer holidays. Suppose he has 53 days in which to accomplish this task and will learn at least one new word a day. Show that during some span of consecutive days, George will learn precisely 15 new words.

## Solution:

Let $b_{i}$ be the number of words George learns on days 1 through $i$ (inclusive) and consider the 106 numbers

$$
b_{1}, b_{2}, \cdots, b_{53}, b_{1}+15, b_{2}+15, \cdots, b_{53}+15
$$

The largest number here is $b_{53}+15=90+15=105$. By the Pigeon Hole Principle, we conclude that two numbers are the same. Since $b_{i}>b_{j}$ if $i>j$, the only way for two to be equal is for two to be equal is for $b_{i}=b_{j}+15$ - that is, $b_{i}-b_{j}=15$ - for some $i$ and $j$. It follows that George learns 15 new words on days $j+1$ trough $i$.

Problem 11: (Section 6.3 Exercise 15) How many seats in a large auditorium have to be occupied to be certain that at least three people seated have the same first and last initials?

## Solution:

By the multiplication rule, there are $26^{2}$ possible pairs of initials. If $n$ seats are occupied, the Pigeon Hole Principle (strong form) says that $\left\lceil\frac{n}{26^{2}}\right\rceil$ people seated will have the same first and last initials. The smallest $n$ such that $\left\lceil\frac{n}{26^{2}}\right\rceil=3$ is $n=2\left(26^{2}\right)+1=1,353$.

Problem 12: (Section 6.3 Exercise 18) A cake is in the shape of a regular hexagon with each of its sides exactly 30 cm long. Seven flowers of icing adorn the top. Show that at least two flowers are not more than 30 cm apart.

## Solution:

Subdivide the hexagon into six equilateral triangles. (1unit $=30 \mathrm{cms}$ ) By the Pigeon Hole Principle, at least two flowers must lie in, or on the boundary of, the same triangle. Hence, these flowers are at most 30 cms apart, the length of one side.

