Assignment 4 Solutions

Problem1: (Section 2.1 Exercise 9) (a.) List all the subsets of the set $\{a, b, c, d\}$ which contain:

(i.) four elements
(ii.) three elements
(iii.) two elements
(iv.) one element
(v.) no elements

Solution:

(i.) $\{a, b, c, d\}$
(ii.) $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$
(iii.) $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$
(iv.) $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$
(v.) $\emptyset$

(b.) How many subsets of $\{a, b, c, d\}$ are there altogether?

Solution:

Altogether there are 16 subsets of $\{a, b, c, d\}$.

Problem 2: (Section 2.1 Exercise 10)
(a.) How many elements are in the power set of the power set of the empty set?

Solution:

If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ is a set containing one element, so its power set contains two elements.

(b.) Suppose A is a set containing one element. How many elements are in $\mathcal{P}(\mathcal{P}(\mathcal{A}))$?

Solution:

$\mathcal{P}(A)$ contains two elements; $\mathcal{P}(\mathcal{P}(\mathcal{A}))$ has four elements.

Problem 3: (Section 2.2 Exercise 12) For $n \in Z$, let $A_n = \{a \in Z | a \leq n\}$. Find each of the following sets.

(b.) $A_3 \cap A_{-3}$

(c.) $A_3 \cap (A_{-3})^c$

(d.) $\cap_{i=0}^{4} A_i$

Solution:

(b.) Since $A_{-3} \subseteq A_3$, $A_3 \cap A_{-3} = A_{-3}$.

(c.) $A_3 \cap (A_{-3})^c = \{a \in Z | -3 < a \leq 3\} = \{-2, -1, 0, 1, 2, 3\}$.

(d.) Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$, we have $\cap_{i=0}^{4} A_i = A_0$

Problem 4: (Section 2.2 Exercise 14)

(b.) Suppose A and B are sets such that $A \cup B = A$. What can you conclude? Why?

Answer: $A \subseteq B$.

Proof:

Assume that B is not a subset of A. That means B contains elements which are not in A.
Then the union $A \cup B$ will contain those elements as well, which would mean that $A \cup B \neq A$.
Therefore B must be a subset of A: $B \subseteq A$.

Problem 5: (Section 2.2 Exercise 17a) Let A, B, and C be subsets of some universal set U.

Prove that $A \cap B \subseteq C$ and $A^c \cap B \subseteq C \rightarrow B \subseteq C$.
Solution:

Let $x \in B$. Certainly x is also in A or in A^c. This suggests cases.
Case 1: If $x \in A$, then $x \in A \cap B$, so $x \in C$.

Case 2: If x does not belong to A, then $x \in A^c \cap B$, so $x \in C$.

In either case, $x \in C$, so $B \subseteq C$.

Problem 6: (Section 2.2 Exercise 18a) Let A, B, and C be sets.

Find a counterexample to the statement $A \cup (B \cap C) = (A \cup B) \cap C$

Solution:

The Venn diagram shown in Fig 2.1 of the text suggests the following counterexample:

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 3, 5, 7\}$. Then $A \cup (B \cap C) = A \cup \{3, 5\} = \{1, 2, 3, 4, 5\}$ whereas $(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6, \} \cap C = \{2, 3, 5\}$

Problem 7: (Section 2.2 Exercise 19) Use the first law of De Morgan to prove the second:

$$(A \cap B)^c = A^c \cup B^c$$

Solution:

Way 1: We use the fact that $(X^c)^c = X$ for any set X.

Let $X = A^c$ and $Y = B^c$. Then $A = X^c$ and $B = Y^c$, so $(A \cap B)^c = [X^c \cap Y^c]^c = [(X \cup Y)^c]^c$ (by the first law of De Morgan)$= X \cup Y = A^c \cup B^c$, as required.

Way 2: (this way we do not use the first law of De Morgan itself, but the proof is similar to the one of the first law of De Morgan)

$$\begin{aligned}(A \cap B)^c &= \{x \in U | \neg(x \in A \wedge x \in B)\} \\ &= \{x \in U | \neg(x \in A) \vee \neg(x \in B)\} \\ &= A^c \cup B^c\end{aligned}$$

Problem 8: (Section 2.3 Exercise 4) With a table like that in Fig 2.2, illustrate a relation on the set {a, b, c, d} which is

(b.) not symmetric and not anti antisymmetric

(d.) transitive

Solution:

(b.)

	a	b	c	d
a	X	X	X	
b		X		
c	X		X	
d				

(d.)

	a	b	c	d
a	X	X	X	X
b		X	X	X
c			X	
d			X	X

Problem 9: (Section 2.3 Exercise 9) Determine whether or not each of the binary re-

lations $\mathcal{R}$ defined on the given sets A are reflexive, symmetric, antisymmetric, or transitive. If a relation has a certain property, prove this is so; otherwise, provide a counterexample to show that it does not.

(d.) $A = R$; $(a, b) \in \mathcal{R}$ if and only if $a^2 = b^2$.

(e.) $A = R$; $(a, b) \in \mathcal{R}$ if and only if $a - b \leq 3$.

(h.) $A = Z$; $\mathcal{R} = \{(x, y) | x + y = 10\}$.

(j.) $A = N$; $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b}$ is an integer.

Solution:

(d.) **Reflexive:** For any $a \in R$, $a^2 = a^2$, so $(a, a) \in \mathcal{R}$.

Symmetric: If $(a, b) \in \mathcal{R}$ then $a^2 = b^2$, so $b^2 = a^2$ which says that $(b, a) \in \mathcal{R}$.

Not antisymmetric: $(1, -1) \in \mathcal{R}$ and $(-1, 1) \in \mathcal{R}$ but $1 \neq -1$.

Transitive: If (a, b) and (b, c) are both in $\mathcal{R}$, then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$ which says $(a, c) \in \mathcal{R}$.

(e.)**Reflexive:** For any $a \in R$, $a - a = 0 \leq 3$ and so $(a, a) \in \mathcal{R}$.

Not Symmetric: For example, $(0, 7) \in \mathcal{R}$ because $0 - 7 = -7 \leq 3$, but $(7, 0) \notin \mathcal{R}$ because $7 - 0 = 7 \not\leq 3$.

Not antisymmetric: $(2, 1) \in \mathcal{R}$ because $2 - 1 = 1 \leq 3$ and $(1, 2) \in \mathcal{R}$ because $1 - 2 = -1 \leq 3$, but $1 \neq 2$.

Not Transitive: $(5, 3) \in \mathcal{R}$ because $5 - 3 = 2 \leq 3$ and $(3, 1) \in \mathcal{R}$ because $3 - 1 = 2 \leq 3$, but $(5, 1) \notin \mathcal{R}$ because $5 - 1 = 4 \not\leq 3$.

(h.)**Not Reflexive:** $(2, 2) \notin \mathcal{R}$ because $2 + 2 \neq 10$.
Symmetric: If $(x, y) \in \mathcal{R}$, then $x + y = 10$, so $y + x = 10$, and hence, $(y, x) \in \mathcal{R}$.

Not antisymmetric: $(6, 4) \in \mathcal{R}$ because $6 + 4 = 10$ and similarly, $(4, 6) \in \mathcal{R}$, but $6 \neq 4$.

Not Transitive: $(6, 4) \in \mathcal{R}$ because $6 + 4 = 10$ and similarly, $(4, 6) \in \mathcal{R}$, but $(6, 6) \notin \mathcal{R}$ because $6 + 6 \neq 10$.

(j.)**Reflexive:** $\frac{a}{a} = 1 \in N$ for any $a \in N$.

Not Symmetric: $(4, 2) \in \mathcal{R}$ but $(2, 4) \notin \mathcal{R}$.

Antisymmetric: If $\frac{a}{b} = n$ and $\frac{b}{a} = m$ are integers then $nm = 1$ so n, $m \in \{\pm\}$. Since a and b are positive, so are n and m. Therefore, $n = m = 1$ and $a = b$.

Transitive: The argument given in Example 24 for Z works the same way for N.

Problem 10: (Section 2.4 Exercise 8) Define $\sim$ on Z by $a \sim b$ if and only if $3a + b$ is a multiple of 4.

(a.) Prove that $\sim$ defines an equivalence relation.

(b.) Find the equivalence class of 0.

(c.) Find the equivalence class of 2.

(d.) Make a guess about the quotient set.

Solution:

(a.) **Reflexive:** For any $a \in Z$, $3a + a = 4a$ is a multiple of 4, so $a \sim a$.

Symmetric: If $a \sim b$, then $3a + b = 4k$ for some integer k. Since $(3a + b) + (3b + a) = 4(a + b)$, we see that $3b + a = 4(a + b) - 4k$ is a multiple of 4, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $3a + b = 4k$ for some integer k and $3b + c = 4l$ for some integer l. Since $4(k + l) = (3a + b) + (3b + c) = (3a + c) + 4b$, we see that $3a + c = 4(k + l) - 4b$ is a multiple of 4 and hence, that $a \sim c$.

(b.)$\overline{0} = \{x \in Z | x \sim 0\} = \{x | 3x = 4k, \text{ for some integer} \, k\}$. Now if $3x = 4k$, k must be a multiple of 3. So $3x = 12l$ for some integer $l \in Z$ and $x = 4l$.

Therefore, $\overline{0} = 4Z = \{0, 4, 8, 12, 16, ...\}$.

(c.)$\overline{2} = \{x \in Z | x \sim 2\} = \{x | 3x + 2 = 4k \text{ for some integer} \, k\} =$
$\{x | 3x = 4k - 2 \text{ for some integer} \, k\}$. Now if $3x = 4k - 2$, then $3x = 3k + k - 2$ and so $k - 2$ is a multiple of 3.
Therefore, $k = 3l + 2$ for some integer l, $3x = 4(3l + 2) - 2 = 12l + 6$ and $x = 4l + 2$.

So $\overline{2} = 4Z + 2 = \{2, 6, 10, 14, 18, ...\}$.

Problem 11: (Section 2.4 Exercise 12) Determine, with reasons, whether or not each of the following defines an equivalence relation on the set A.

(b.) A is the set of all circles in the plane; $a \sim b$ if and only if a and b have the same center.

(c.) A is the set of all straight lines in the plane; $a \sim b$ if and only if a is parallel to b.

(d.) A is the set of all lines in the plane; $a \sim b$ if and only if a is perpendicular to b.

Solution:

(b.) Yes, this is an **Equivalence Relation**.

Reflexive: If a is a circle, then $a \sim a$ because a has the same center as itself.

Symmetric: Assume $a \sim b$. Then a and b have the same center. Thus, b and a have the same center, so $b \sim a$.

Transitive: Assume $a \sim b$ and $b \sim c$. Then a and b have the same center and b and c have the same center, so a and c have the same center. Thus $a \sim c$.

(c.) Yes, this is an **Equivalence Relation**.

Reflexive: If a is a line, then a is parallel to itself, so $a \sim a$.

Symmetric: If $a \sim b$, then a is parallel to b. Thus, b is parallel to a. Hence, $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then a is parallel to b and b is parallel to c, so a is parallel to c. Thus $a \sim c$.

(d.) No, this is **Not** an equivalence relation. The reflexive property does not hold because no line is perpendicular to itself. Neither is this relation transitive; if l_1 is perpendicular to l_2 and l_2 is perpendicular to l_3, then l_1 and l_3 are **parallel**, not perpendicular to one another.