Math 3210 **Due Tue Oct 6** **Assignment #3**

Definition 1 Complex Derivative.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}, \quad z \in C.$$

Definition 2 Continuity of complex function at point z_0.

$$\lim_{z \to z_0} f(z) = f(z_0).$$

1. Show that $\lim_{z\to 0}(z/\bar{z})^2$ does not exist by comparing limits along different directions towards the origin in the complex plane.

2. Show by definitions that $f(z) = Re(z)$ is a continuous but not differentiable function at any point z in the complex plane.

3. Let $f(z) = 3|z|^2 + 5z - 6$. Find using the definition $f'(0)$. Does the derivative exist at any other point besides the origin?

4. Rewrite the limit in an equivalent form avoiding ∞'s. Explain why the statement is true.
   (a) $\lim_{z\to\infty} \frac{2z^2}{(3z+1)^2} = 2/9$; (b) $\lim_{z\to\infty} \frac{(2z+1)^3}{(1+100z)} = \infty$;

5. Find complex derivative using differentiation rules (a) $z^5(-iz^4 - 2)^7$; (b) $\left(\dfrac{z^2 - 2i}{z^3 + 10}\right)^3$

6. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic. Derive formula
   $$f'(z) = -i(u'_\theta + iv'_\theta)/z$$
   from the formula $f'(z) = e^{-i\theta}(u'_r + iv'_r)$ using Cauchy-Riemann equations in polar form.

7. Use Cauchy-Riemann equation in the appropriate form to determine where in the complex plane the following function is differentiable. Find the derivative if it exists.
   (a) $f(z) = (2\bar{z} - 1)^2$; (b) $f(z) = iRe(z) - Im(z)$; (c) $f(z) = z^{-2}$, $z \neq 0$; (d) $e^x e^{-iy}$;
   (e) $e^{-\theta}(\cos(\ln r) + i\sin(\ln r))$; (f) $r^2 \sin 2\theta - ir^2 \cos 2\theta$; (g) $\bar{z}^2 - z^2$.

8. **Extra Points Problem** Let $f'(0) = 1$ and $f(0) = 1$. Prove using the definitions that
   (a) $f(z)$ is continuous at $z = 0$; (b) there exists $r > 0$ such that $f(z) \neq 0$ for $|z| < r$.

9. **Extra Points Problem** What is the flaw in the following argument?
   $$e^{i\theta} = \left(e^{i\theta}\right)^{2\pi/2\pi} = \left(e^{2\pi i}\right)^{\theta/2\pi} = 1^{\theta/2\pi} = 1.$$