Math 3210 **Answers** **Assignment #7**

1. Write the statement of the Cauchy-Goursat Theorem. Read the proof of the Cauchy-Goursat Theorem (p.144-149). Write down the main steps of the proof.

   This question was not marked. The Theorem and the prove are in the book and in the lecture notes. Come to discuss it you have questions/remarks

2. Write the statement of the Cauchy Integral Formula. Read and write down the proof of the Cauchy Integral Formula.

   This question was not marked. The Theorem and the prove are in the book and in the lecture notes. Come to discuss it you have questions/remarks

3. Evaluate the contour integral. Assume each contour to be positively oriented. Use Cauchy-Goursat Theorem or Cauchy Integral formula where applicable.

   (a) $\oint_{|z|=1} \tan z\, dz = 0$

   (b) $\oint_{|z|=4} \dfrac{z}{z+3}\, dz = -6\pi i$

   (c) $\oint_{|z|=2} \dfrac{z}{z+3}\, dz = 0$

   (d) $\oint_{|z|=3} \dfrac{z}{(z+2)(z+4)}\, dz = -2\pi i$

   (e) $\oint_{|z|=2} \cosh z\, dz = 0$

   (f) $\oint_{|z|=2} \dfrac{\cosh z}{z}\, dz = 2\pi i$

   (g) $\oint_{|z|=2} \mathrm{Log}(z+5)\, dz = 0$

   (h) $\oint_{|z|=2} (z-1)^{-1}\mathrm{Log}(z+5)\, dz = 2\pi i \ln 6$

4. Let contour C be a pentagon with vertices at $-1; \pm 3i; 2 \pm 2i$. For all integer n find value of the integral
$$\int_C (z-1-i)^n dz.$$
   This is 0 for all $n \neq -1$, and $\pm 2\pi i$ for $n = -1$. The sign depends on thecontour orientation.

5. Which of the following is true? Give a proof or a counterexample.

   A: Let C be a closed simple contour. If $\int_C f(z)\, dz = 0$ then $f(z)$ is analytic in the interior of C?

   This is FALSE. Examples are $\oint_{|z|=R} z^{-2} dz = 0$, while $f(z) = z^{-2}$ is not continuous at $z = 0$, so it is not analytic inside the circle $|z| < R$. Another example is $\oint_{|z|=R}(|z|^2 - R^2)\, dz = 0$ (here $R =$const), but $f(z) = |z|^2 - R^2$ is not analytic in $|z| < R$.

   B: Let $f(z)$ be continuous and $\int_C f(z)\, dz = 0$ for any closed simple contour C. Then $f(c)$ is analytic.

   This is TRUE, and is in fact statement of Morera Theorem. The proof follows from the fact that such a function must have an antiderivative and therefore is analytic.

6. What is the geometrical meaning of expression $-i\int_C \bar{z}\,dz$, here C is a simple closed contour.

   This is the double area enclused by the contour, $2R_C$. Proof uses Green's Therem $\int_C Pdx + Qdy = \int\int_{R_C}(Q_x - P_y)dA$ and is as follows

$$\int_C \bar{z}\,dz = \int_C (x-iy)(dx+idy) = \int_C xdx + ydy + i\int_C xdy - ydx = 0 + 2i\int\int_{R_C} dA.$$

7. Write down and simplify the integral of $\exp(-z^2)$ along each side of a rectangle with vertices at $\pm a; \pm a + ib$, $b > 0$. Use Cauchy-Goursat Theorem to show that

$$\int_0^a e^{-x^2}\cos(2bx)\,dx = e^{-b^2}\int_0^a e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)\,dy.$$

   Use this result to evaluate $\int_{-\infty}^{\infty} e^{-x^2}\cos(2bx)\,dx$

   Solution. Note $e^{-z^2} = e^{-(x^2-y^2)}e^{-2xyi}$.

   Side 1: $y = 0, -a \le x \le a$, then $e^{-z^2} = e^{-x^2}$.

   Side 2: $y = b, -a \le x \le a$, then $e^{-z^2} = e^{-x^2}e^{b^2}e^{-2xbi}$.

   Side 3: $x = a, 0 \le y \le b$, then $e^{-z^2} = e^{-a^2}e^{y^2}e^{-2ayi}$.

   Side 4: $x = -a, 0 \le y \le b$, then $e^{-z^2} = e^{-a^2}e^{y^2}e^{2ayi}$.

   The function e^{-z^2} is analitic in the rectangular contour, co the sum of the integrals along sides is 0. This gives (using positive orientation)

$$\int_{-a}^a e^{-x^2}dx - \int_{-a}^a e^{-x^2}e^{b^2}e^{-2xbi}dx + \int_0^b e^{-a^2}e^{y^2}e^{-2ayi}idy - \int_0^b e^{-a^2}e^{y^2}e^{2ayi}idy = 0.$$

   Using $\int_{-a}^a e^{-x^2}\sin(2xb) = 0$ and $\int_{-a}^a e^{-x^2}\cos(2xb) = 2\int_0^a e^{-x^2}\cos(2xb)$, and multiplying the whole equation by $e^{-b^2}/2$ we obtain the first formula.

   Now we take the limit $a \to \infty$ and use $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. Also $|\int_0^b e^{y^2}\sin(2ay)dy| \le \int_0^b e^{y^2}dy$, so the second term gives 0 in the limit. Thus

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2bx)\,dx = \sqrt{\pi}e^{-b^2}$$

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