Math 3210 **Answers** **Assignment #6**

1. Give an example (with explanaition) of

   (a) simple closed non-differentiable curve;
   A triange or a square: no self-intersections, closed, not differentible at the corners.

   (b) differentiable but not smooth arc;
   An arc given by $z(t) = 1 + i\sin t$, $0 \le t \le \pi$. Then $z'(\pi/2) = 0$ so is not smooth.

   (c) Jordan curve.
   This is a simple closed curve, for example a circle.

2. Let
$$z_1(t) = R_0 e^{it}, \quad z_2(s) = \sqrt{R_0^2 - s^2} + is, \quad z_3(q) = q + i\sqrt{R_0^2 - q^2},$$
where $R_0 > 0$ is a constant and t, s, q are real parameters. When the interval for t is given, $z_1(t)$ describes a curve in the complex plane. Give an alternative description of the same curve in terms of either $z_2(s)$ or $z_3(q)$ (or both) and find corresponding interval(s) for s and q if

   (a) $\pi/4 < t < 2\pi/3$.
   An alternative description is $z_3(-q)$, where $q \in [-R_0/\sqrt{2}, R_0/2]$.

   (b) $-\pi < t < -\pi/3$
   An alternative description is $\bar{z}_3(q)$, where $q \in [-R_0, R_0/2]$.

   (c) $3\pi/4 < t < 5\pi/4$
   An alternative description is $-z_2(s)$, where $s \in [-R_0/\sqrt{2}, R_0/\sqrt{2}]$.

   (d) $-\pi/2 < t < \pi$
   An alternative description is the union of $z_2(s)$, where $s \in [-R_0, R_0]$ and $z_3(-q)$, where $q \in [0, R_0]$.

3. Evaluate contour integral along curve $C = \{z(t), a \le t \le b\}$.

   (a) $\int_C z^{-2}\, dz$, $z(t) = 5e^{2it}$, $0 \le t \le \pi/4$.
   Answer: $(1+i)/5$.

   (b) $\int_C \bar{z}^2\, dz$, $z(t) = e^{-it}$, $0 \le t \le 2\pi$.
   Answer: 0.

   (c) $\int_C \bar{z}^{-n}\, dz$, $z(t) = e^{ikt}$, $0 \le t \le 2\pi/k$, where n and $k \ne 0$ are integers.
   Answer: 0 for all $n \ne -1$ and $2\pi i$ for $n = -1$.

   (d) $\displaystyle\int_C \frac{3+z}{z}\, dz$, $z(t) = \sqrt{9 - t^2} + it, 0 \le t \le 3$
   Here you may want to change the contour description to $z(s) = 3e^{is}$, where $0 \le s \le \pi/2$.
   Then contour integration gives $3i(\pi + 2)/2 - 3$.

   (e) $\int 2xy + i(y^2 - x^2)dz$ with respect to triangular contour with vertices at points $z = \pm i$ and $z = -1$.
   Integrals with respect to each side of the triangle are : $-3/2$, $(1+i)/3$ and $(1-i)/3$, which sums up to zero.
   The answer also follows from the Cauchy-Goursat theorem because the integrant is iz^2 which is analytic.

4. Do the following without evaluating the integral.

   (a) Let $C = \{2e^{it}, 0 \le t \le \pi/2\}$. Show that $|\int_C (z^4-1)^{-1}\,dz| \le \pi/7$.
   Here $L = \pi$ and $M = (2^4-1)^{-1} = 15$. Thus
   $|\int_C (z^4-1)^{-1}\,dz| \le \pi/15 < \pi/7$.

   (b) Let C be a line segment from $z = -i$ to $z = 1$. Show that $|\int_C z^{-6}\,dz| \le 8\sqrt{2}$
   Note that point $(1/\sqrt{2}, -1/\sqrt{2})$ is the closest point to the origin among all point from the segment of integration. Thus the max value of the integrant is reached there and is $M = 8$. The length $L = \sqrt{2}$. This gives the estimate.

   (c) Let C be a circle of radius R. Show that

$$|\int_C \frac{\mathrm{Log} z}{z^4}\,dz| \le 2\pi\frac{\ln R + \pi}{R^3}.$$

   By definition, $Logz = \ln|z| + i\Theta$, $-\pi < \Theta \le \pi$. Thus $M = \dfrac{\ln R + \pi}{R^4}$ and $L = 2\pi R$.This gives the estimate.

5. **Extra Points Problem** Show that condition

$$\frac{\partial f(z, \bar{z})}{\partial \bar{z}} = 0$$

is equivalent to the Cauchy-Riemann equations.

Solution. Let $f(x,y) = u(x,y) + iv(x,y)$. Using $x = (z+\bar{z})/2$ and $y = i(\bar{z}-z)/2$ and the chain rule we obtain

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + iv_x) + \frac{i}{2}(u_y + iv_y) = \frac{u_x - v_y}{2} + i\frac{v_x + u_y}{2}.$$

Recall the Cauchy-Riemann condition $u_x = v_y$ and $v_x = -u_y$, so it gives desired statement.