Math 3210 **Due Thur Oct 14** **Assignment #4**

1. Show that function $v(x, y)$ is harmonic and find the function $u(x, y)$ of which it is harmonic conjugate.

   (a) $v = 2y - 3x^2y + y^3$
   Answer: $v_{xx} + v_{yy} = -6y + 6y = 0$ thus it is harmonic.
   $u(x, y) = 2x - x^3 + 3xy^2 + C$. Check: $u_x = v_y = 2 - 3x^2 + 3y^2$ and $u_y = -v_x = 6xy$.

   (b) $v = \cos y \cosh x$
   Answer: $u(x, y) = -siny \sinh x + C$

   (c) $v = x/(x^2 + y^2)$
   Answer: $u(x, y) = y/(x^2 + y^2) + C$

2. Show that $F(z)$ is analytic in D if and only if $-iF(z)$ is analytic there.

   Answer: By def $F(z)$ is analitic in D if it has derivative at each point together with its open vicinity in D. Multiplication by a constant will not effect this property.

   Another way to prove it is to use Cauchy-Riemannn equations. Assuming that real and imaginary parts of F obey them, show that real and imaginary parts of iF will obey them as well.

3. **Lemma** Let $f(z) = u(x, y) + iv(x, y)$ be analytic. Let curves $u(x, y) = c_1$ and $v(x, y) = c_2$ intersect at point z_0 and $f'(z_0) \neq 0$. Then the lines tangent to the curves at z_0 are perpendicular.

   (a) Demonstrate the Lemma for $f(z) = z^{-1}$, $z \neq 0$;
   Answer: $f(z) = z^{-1} = \bar{z}/|z|^2$. Then $u = x/(x^2 + y^2)$ and $v = -y/(x^2 + y^2)$. Equation $u(x, y) = c_1$ becomes a circle shifted from the origin along the x axis. Equation $v(x, y) = c_2$ becomes a circle shifted from the origin along the y axis. Such two circles, if intersect, form the right angle.

   (b) Prove the Lemma. Hint: regard y as a function of x and differtiate equations of the curves w.r.t. x. Then, find product of slopes of tangent lines at the intersection point. Use C.-R. equations to show that the product is -1. Make conclusion.
   Answer: $u(x, y(x)) = c_1$; so $u_x + u_y y'(x) = 0$; thus $y'(x) = -u_x/u_y = k_1$.
   $v(x, y(x)) = c_2$; so $v_x + v_y y'(x) = 0$; thus $y'(x) = -v_x/v_y = k_2$.
   The product of the slopes at the point of intersection x is $k_1k_2 = u_xv_x/(u_yv_y)$. But f is analytic, so $u_x = v_y$ and $u_y = -v_x$, so $k_1k_2 = -1$. Thus the lines are perpendicular.

4. Consider functions

   $f_1(z) = \sqrt{r}e^{i\theta/2}$, $r > 0$, $0 < \theta < \pi$;
   $f_2(z) = \sqrt{r}e^{i\theta/2}$, $r > 0$, $\pi/2 < \theta < 2\pi$;
   $f_3(z) = \sqrt{r}e^{i\theta/2}$, $r > 0$, $3\pi/2 < \theta < 5\pi/2$;

   Show that f_2 is analytic continuation of f_1, f_3 is analytic continuation of f_2, but $f_3 \neq f_1$ on their common domain.

   Answer: All three functions are analytic on their domains. $f_1 = f_2$ on the intersection of their domains $\pi/2 < \theta < \pi$. Thus f_2 is analytic continuation of f_1.

   Likewise, $f_2 = f_3$ on $3\pi/2 < \theta < 2\pi$, so f_3 is analytic continuation of f_2, and by transitivity, f_3 is analytic continuation of f_1.

   Nevertheless, $f_3(z) = -f_1(z)$ for any point from the first quadrant. For example, $z = 1 + i$, $f_3(1 + i) = f_3(\sqrt{2}e^{i(\pi/4+2\pi)}) = -2^{1/4}e^{i\pi/8}$, but $f_1(1 + i) = f_1(\sqrt{2}e^{i\pi/4}) = 2^{1/4}e^{i\pi/8}$.

5. Check if the following function satisfy $f(\bar{z}) = f(\bar{z})$ by two ways: directly and using reflection principle.

$$a) f(z) = z^3 \text{ yes} \qquad b) f(z) = z^3(1+i) \text{ no}$$

$$c) f(z) = e^z \text{ yes} \qquad d) f(z) = e^{iz} \text{ no}$$

6. Evaluate

   (a) $\exp(\frac{2+i\pi}{4}) = \sqrt{e/2}(1+i)$
   (b) $(\exp(z^4))' = 4ei$ at $z = -i$
   (c) $\log i = i(\pi/2 + 2\pi k)$
   (d) $\log e = 1 + (2\pi k)i$
   (e) $\log(1+\sqrt{3}i) = \ln 2 + i(\pi/3 + 2\pi k)$

7. Solve for z

   (a) $\log z = i\pi/2$ *Answer:* $z = i$.
   (b) $\tan z = 2i$ *Answer:* $z = \pi/2 + k\pi + i(\ln 3)/2$.
   (c) $\sinh z = i$ *Answer:* $z = i(\pi/2 + 2k\pi)$
   (d) $\sinh(\cos z) = 0$ *Answer:* $z = \pi/2 + 2k\pi - 2Log(n\pi + \sqrt{n^2\pi^2 + 1})$

8. Show that $\text{Log}(1+i)^2 = 2\text{Log}(1+i)$, but $\text{Log}(-1+i)^2 \neq 2\text{Log}(-1+i)$.
   Answer: $Log(1+i)^2 = Log(2i) = \ln 2 + i\pi/2$
   $2Log(1+i) = 2(\ln\sqrt{2} + i\pi/4) = \ln 2 + i\pi/2 = Log(1+i)^2$;
   $Log(-1+i)^2 = Log(-2i) = \ln 2 - i\pi/2$
   $2Log(-1+i) = 2(\ln\sqrt{2} + i3\pi/4) = \ln 2 + i3\pi/2 \neq Log(-1+i)^2$;

9. **Extra Points Problem** Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic. Find 2nd order partial differetial equation for function $u(r,\theta)$. How does it relate to the Laplace equation we considered before?
   Answer: $r^2u_{rr} + ru_r + u_{\theta\theta} = 0$. This is Laplace equation in polar coordinates.