Math 3210 **Solutions** **Assignment #3**

Definition 1 Complex Derivative.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}, \quad z \in C.$$

Definition 2 Continuity of complex function at point z_0.

$$\lim_{z \to z_0} f(z) = f(z_0).$$

1. Using definition, find derivative of z^n, for

   (a) any positive integer n;

   Solution. Use Definition 1 $f(z) = z^n$ and binomial expansion for $n \geq 1$

   $$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

   with $a = z$ and $b = \Delta z$.

   (b) any negative integer n, if $z \neq 0$.

   Solution. Use Definition 1 for $f(z) = z^n$ and binomial expansion for $n \leq -1$

   $$(a+b)^n = a^n + \sum_{k=1}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} a^{n-k} b^k$$

   with $a = z$ and $b = \Delta z$.

2. Show using definitions only that $f(z) = \bar{z}$ is continuous but not differentiable at any point of the complex plane.

   Solution.
   $$\lim_{z \to z_0} \bar{z} = \lim_{(x,y) \to (x_0,y_0)} (x - iy) = x_0 - iy_0 = \bar{z}_0.$$
   Thus the function is continuous by Def 2. Now,

   $$\lim_{\Delta z \to 0} \frac{(\bar{z} + \bar{\Delta z}) - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{\Delta z}}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

   and this limit does not exist since its value depends upon the path. Say, path $\Delta y = 0, \Delta x \to 0$ gives 1, while path $\Delta x = 0, \Delta y \to 0$ gives -1. Thus the function is not differenciable.

3. Find complex derivative using differentiation rules

   (a) $(-4z^3 + i)^9$

   Solution. By combination of the chain and power rules we get

   $$\frac{d}{dz}(-4z^3 + i)^9 = -108z^2(-4z^3 + i)^8.$$

(b) $\left(\dfrac{iz-1}{z^2+1}\right)^4$

Solution. One may simplify $\left(\dfrac{iz-1}{z^2+1}\right)^4 = \left(\dfrac{1}{iz+1}\right)^4$.

Then the derivative is

$$\frac{d}{dz}\left(\frac{1}{iz+1}\right)^4 = \frac{-4i}{(iz+1)^5}.$$

4. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic. Derive formula

$$f'(z) = -i(u'_\theta + iv'_\theta)/z$$

from the formula $f'(z) = e^{-i\theta}(u'_r + iv'_r)$ using Cauchy-Riemann equations in polar form.

Solution.

$$f'(z) = e^{-i\theta}(u'_r + iv'_r) = e^{-i\theta}(\frac{v'_\theta}{r} - i\frac{u'_\theta}{r}) = \frac{-i}{re^{i\theta}}(u'_\theta + iv'_\theta) = \frac{-i(u'_\theta + iv'_\theta)}{z}.$$

5. Use Cauchy-Riemann equation in the appropriate form to determine where in the complex plane the following function is analytic. Find the derivative if it exists.

General remark. In this problem we use the following. Let

$$f(z) = u(x,y) + iv(x,y) = U(r,\theta) + iV(r,\theta).$$

Cauchy-Riemann (C.R.) equations are

$$u_x = v_y, \quad u_y = -v_x,$$

or in polar form

$$rU_r = V_\theta, \quad rV_r = -U_\theta.$$

Now, if C.R. condition holds at z and all partial derivatives are continuous at z then the derivative $f'(z)$ exista and is equal to

$$f'(z) = u_x + iv_x = e^{-\theta}(U_r + iV_r).$$

Now, function is analitic in an open set if it has a derivative in the open set.

Note that if a function has derivative at just one point or just at a line in the complex plane, it is not analitic because a point or a line are not open sets in the plane.

(a) $f(z) = (\bar{z}+2)^2$

Solution. Here $u = x^2 - y^2 + 4x + 4$ and $v = -2xy - 4y$. So, $u_x = 2x + 4$, $v_y = -2x - 4$, $u_y = -2y$, $v_x = -2y$. The C.R. condition holds only at one point $x = y = 0$. So the function has derivative only at one point. Thus it is not analytic in the whole complex plane.

(b) $f(z) = \mathrm{Im} z$

Solution. Here $u = 0$, $v = y$. C.R. condition does not hold so the function is not analytic.

(c) $f(z) = z^{-3}$, $z \neq 0$

Solution. Here it is better to use the polar form. Let $z = re^{i\theta}$. Then $f = r^{-3}e^{-3i\theta}$. So, $u = r^{-3}\cos(3\theta)$, $v = -r^{-3}\sin(3\theta)$. C.R. in polar form holds so the function in analytic in whole its domain i.e. in the complex plane without the origin. The derivatine is

$$f' = e^{-i\theta}(-3r^{-4}\cos(3\theta) - i3r^{-4}\sin(3\theta)) = -3z^{-4}.$$

(d) $\sqrt{r}e^{i\theta/2}$, $r > 0$, $-\pi < \theta < \pi$

Solution. Here $u = \sqrt{r}\cos(\theta/2)$, $v = \sqrt{r}\sin(\theta/2)$. The C.R. in polar form holds. The function is analitic in the domain specified.
The derivatine is

$$f' = e^{-i\theta}(1/2r^{-1/2}\cos(\theta/2) + i(1/2)r^{-1/2}\sin(\theta/2)) = 1/2z^{-1/2}.$$

(e) $2x + ixy^2$

Solution. Here $u = x$, $v = xy^2$. C.R. does not hold. The function is not analitic.

(f) $e^{\bar{z}}$

Solution. Here $u = e^x \cos y$ and $v = -e^x \sin y$. C.R. does not hold so the function is not analitic.

(g) $r\cos\theta + ir$

Solution. Here. $u = r\cos\theta$, $v = r$. So $u_r = \cos\theta$, $v_r = 1$, $u_\theta = -r\sin\theta$, $v_\theta = 0$.
C.R. condition in polar holds for $\theta = \pi/2 + 2\pi k$.
Since the ray $\theta = \pi/2$ is not an open set in the copmlex plane, the function is not alalitic in the whole plane.

(h) $(e^{2x}\cos 2y + ie^{2x}\sin 2y)^{-1}$

*Solution.*Rewrite as $e^{-2x-i2y} = e^{-2z} = r^{-2}e^{-2i\theta}$. Then use C.R. in polar coordinates to check that the function is analitic in the whole plane but the origin, and its derivative is $-2e^{-2z} = -2(e^{-2x}\cos 2y - ie^{-2x}\sin 2y)$.
P.S. Same result can be obtained without the switch to polar representation.

(i) $r^4\sin 4\theta - ir^4\cos 4\theta$

Solution. Rewrite as $r^4\sin 4\theta - ir^4\cos 4\theta = -ir^4e^{4i\theta} = -iz^4$. This function is analitic and its derivative is $-4iz^3$.
The same result must follow from direct calculations using C.R. equations.

(j) $\bar{z}^3$

Solution. Use $\bar{z} = re^{-i\theta}$. So $\bar{z}^3 = r^3 e^{-3i\theta}$. Then $u = r^3 \cos 3\theta$ and $v = -r^3 \sin 3\theta$. C.R. in polar does not hold, so the function is not analytic.

6. **Extra Points Problem** What is the flaw in the following argument?

$$e^{i\theta} = \left(e^{i\theta}\right)^{2\pi/2\pi} = \left(e^{2\pi i}\right)^{\theta/2\pi} = 1^{\theta/2\pi} = 1.$$