Math 3210 **Answers** **Assignment #10**

1. Write an essay "[Interesting, Amazing, Weird, Funny] *Facts Known About Analytic Functions.*" (about 2 pages)

   This question will not be marked but you may be allowed to use it during the Final Exam.

2. Pretend that you are Cauchy/Riemann/Liouville/Gauss/Euler/Argand/De Moivre/Hamilton/Morera/Laplace/Mkondra/Goursat/Taylor/Laurent and type your favorite formula on my Message Board. http://www.math.mun.ca/ mkondra/okno/index.php Please, select Math 3210

3. Compose a True/False or multi-choise question about analytic functions and solve it. If I like it I may use it for our final exam.

4. Find the Residue $\mathrm{Res}_{z=a} f(z)$ for given function $f(z)$ at point a.

   (a) $f(z) = z^7 \cos(z^{-2})$, $a = 0$.

   Answer: $\mathrm{Res}_{z=0} z^7 \cos(z^{-2}) = \dfrac{1}{4!} = \dfrac{1}{24}$

   use power series for $\cos w = 1 - \dfrac{w^2}{2} + \dfrac{w^4}{4!} - \ldots$ and sub $w = z^{-2}$, then multiply through by z^7.

   (b) $f(z) = z^{-3} \cot z$, $a = 0$.

   Answer: $\mathrm{Res}_{z=0} z^{-3} \cot z = 0$

   use Laurent series for $\cot z = \dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} - \ldots$ (all odd powers of z), then multiply through by z^{-3}.

   (c) $f(z) = \dfrac{\sinh z}{z^4(1-z^2)}$, $a = 0$.

   Answer: $\mathrm{Res}_{z=0} \dfrac{\sinh z}{z^4(1-z^2)} = 1 + \dfrac{1}{3!} = \dfrac{7}{6}$

   use series for $\sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \ldots$ and $(1-z^2)^{-1} = 1 + z^2 + z^4 + \ldots$.

   (d) $f(z) = \dfrac{z^{1/4}}{z+2}$, $a = -2$, $0 < \arg z < 2\pi$.

   Answer: $\mathrm{Res}_{z=-2} \dfrac{z^{1/4}}{z+2} = (-2)^{1/4} = \dfrac{1+i}{2^{1/4}}$

   (e) $f(z) = \dfrac{Log z}{(z^2+1)^2}$, $a = i$.

   Answer: $\mathrm{Res}_{z=i} \dfrac{Log z}{(z^2+1)^2} = g'(i) = \dfrac{i}{4} + \dfrac{\pi}{8}$.

   Here $g(z) = \dfrac{Log z}{(z+i)^2}$ and $\dfrac{Log z}{(z^2+1)^2} = \dfrac{g(z)}{(z-i)^2}$.

5. Evaluate each integral, if **contours have negative orientation**.

   General remark: in (c),(e),(f),(g) the Single Residue technique gives faster solution. Note that all of them have more then one isolated singular point inside the contour.

   (a) $\displaystyle\oint_{|z|=2} \frac{e^z\,dz}{z(z-1)} = 2\pi i(1-e).$

   (b) $\displaystyle\oint_{|z|=2} \frac{dz}{z^3(z+3)} = -\frac{2\pi i}{27}.$

   (c) $\displaystyle\oint_{|z+2|=3} \frac{dz}{z^3(z+3)} = 0.$

   (d) $\displaystyle\oint_{|z|=2} \tan z\,dz = 4\pi i.$

   (e) $\displaystyle\oint_{|z|=4} \frac{dz}{z(z+1)(z+2)(z+3)} = 0.$

   (f) $\displaystyle\oint_{|z|=2} \frac{z^3(1-3z)\,dz}{(1+z)(1+2z^4)} = 3\pi i.$

   (g) $\displaystyle\oint_{|z|=2} \frac{z^5\,dz}{1-z^3} = 2\pi i.$

   (h) $\displaystyle\oint_{|z|=2} z^3 e^{1/z}\,dz = -\frac{\pi i}{12}.$

6. Is it true or false that for any $R > 0$

$$\oint_{|z|=R} \exp(z+z^{-1})\,dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

   Answer: true, if the contour is positively oriented.

   expand $e^z = 1+z+z^2/2!+\cdots z^n/n!+\cdots$ and use $\operatorname{Res}_{z=0} z^n e^{1/z} = 1/(n+1)!$ to evaluate the integral.

7. Let C_N be a positively oriented square centered at the origin with sides parallel to the coordinate axes, and of size $(2N+1)\pi$. Show that

$$\oint_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

   Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

   Answer: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$

   Use $\displaystyle\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \cdots$ then $\displaystyle\operatorname{Res}_{z=0} \frac{1}{z^2\sin z} = \frac{1}{6}$. Also $\displaystyle\operatorname{Res}_{z=\pi n} \frac{1}{z^2 \sin z} = \frac{1}{(\pi n)^2 \cos(\pi n)}$ for $n \neq 0$.

   Take limit $N \to \infty$ and observe that then $\displaystyle\oint_{C_N} \frac{dz}{z^2 \sin z} \to 0$. Thus the RHS is zero as well.